# 2.1 - Definition and Examples of Categories

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(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

The first chapter generalized the notion of an algebraic structure, and dealt with homomorphisms in between them. Now we do something even weirder: we generalize the notion of a homomorphism, not necessarily between sets!

A category consists of a community of objects and morphisms between them. It does not have any trivial structure we have previously dealt with. However, given a category, we can define many things reasonably, and give proofs which focus on morphisms. It just comes to a challenge that we cannot treat the objects like sets [until Section 9 comes along].

This is an ubiquitous concept in mathematics. It starts out involving algebraic structures, but then changes to entirely different structures, like the ones in the later chapters. What's more awkward is that categories have morphisms of their own [Section 3], and these morphisms have *their* own morphisms!

To see what morphisms would look like, recall the basic properties of homomorphisms of $\mathcal{V}(S)$ algebras.

To begin with, the domain and codomain of a homomorphism are intrinsic, even though surjectification is possible. [For example, the set map $\{0\} \to \mathbb{Z}$ sending 0 to 2 is different from the map $\{0\} \to \mathbb{R}$ sending 0 to 2.] This is a rule which prevents any confusion in category theory.

The second thing to realize is that if $f : A \to B$ and $g : B \to C$ are homomorphisms, so is $gf : A \to C$ [Theorem 1.4(1)]. Reread the statement in the proof if you don't remember why.

Next, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(hg)f = h(gf)$, because both send $x \in A$ to $h(g(f(x))) \in D$. It is well-known that composition of functions is associative, no matter what the functions are.

The final thing about homomorphisms is that $1_A : A \to A$ is a homomorphism [Theorem 1.4(2)], and whenever $f : A \to B$ and $g : B \to A$, clearly $f1_A = f$ and $1_A g = g$.

Abstracting the properties just gone over:

**DEFINITION**

*A **category** is a mathematical object* $\mathbf{C}$ *with new structure given by the following:*

(1) $ob(\mathbf{C})$ *is a class, whose elements are called the **objects** of* $\mathbf{C}$.

(2) *For each* $A, B \in ob(\mathbf{C})$, $\hom_{\mathbf{C}}(A, B)$ *[or* $\hom(A, B)$ *if* $\mathbf{C}$ *is clearly under discussion] is a set whose elements are called **morphisms** from* $A$ *to* $B$*. One writes* $f : A \to B$ *for* $f \in \hom(A, B)$.

(3) *If* $(A, B) \neq (A', B')$, $\hom(A, B)$ *and* $\hom(A', B')$ *are disjoint.*

(4) *Whenever* $f : A \to B$ *and* $g : B \to C$*, the **composite function*** $gf$ *is some morphism* $A \to C$*. Stated otherwise, for each* $A, B, C \in ob(\mathbf{C})$*, a map* $\hom(B, C) \times \hom(A, B) \to \hom(A, C)$ *is equipped.*

(5) [associativity] *When* $f : A \to B$, $g : B \to C$ *and* $h : C \to D$, $(hg)f = h(gf)$. *As usual, we simplify this to* $hgf$.

(6) [identity] *For each* $A \in ob(\mathbf{C})$, *there is a morphism* $1_A : A \to A$ *such that whenever* $f : A \to B$ *and* $g : B \to A$, $f1_A = f$ *and* $1_A g = g$. $1_A$ *is called the* ***identity morphism*** *on* $A$.

**REMARKS** Condition (3) is useful, but it is not very necessary. Whatever sets the $\hom(A, B)$'s are, their elements could be tagged indicating where they are, making the sets disjoint. And the identity morphism $1_A$ is unique, because if $1'_A : A \to A$ also satisfies the condition, $1_A = 1_A 1'_A = 1'_A$.

Since this definition is hard to understand, examples would surely help.

**EXAMPLES**

1. A variety $\mathcal{V}(S)$ becomes a category $\mathbf{V}(S)$ with $\mathrm{ob}(\mathbf{V}(S))$ the class of $\mathcal{V}(S)$ algebras, and $\hom(A, B)$ the set of homomorphisms $A \to B$, where composition of morphisms is the usual function composition and $1_A : A \to A$ is the identity map on $A$. [Note that $\hom(A, B)$ may be empty.] In particular, varieties we know already yield the categories **Mon** [monoids], **Grp** [groups], **Ab** [abelian groups], **Ring** [rings], **Rng** [rngs], **Rinv** [rings with involution], $R-$**mod** [left $R$-modules with $R$ a given ring], $M-$**act** [left $M$-actions], and lots more ... and, of course, **Set**, the category of sets.

2. In fact, using Exercise 5 of Section 1.11, one can form the category of all varieties **Var** where the morphisms are takeoffs. This category is quite complicated, because of the possible difficulty in verifying the axioms.

3. The integral domains form a category **Dom** where homomorphisms are the usual ring homomorphisms. Note, however, that the integral domains *don't* form a variety.

4. The fields form a category **Field** where homomorphisms are the usual ring homomorphisms. Note, by the way, that all of the homomorphisms are injective! [Exercise 1] Similarly, if $F$ is a particular field, one could form the category $F$**-Ext** of extension fields of $F$, where only homomorphisms that send every element of $F$ to itself are admitted.

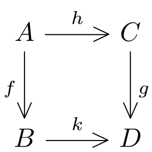
5. A category $\mathbf{C}$ is said to be **discrete** provided $\hom(A, B) = \emptyset$ when $A \neq B$ and $\hom(A, A) = \{1_A\}$. Discrete categories can be identified purely with their objects.

6. Notice that if $A$ is an object in $\mathbf{C}$, $\hom(A, A)$ is a monoid with the categorical structure. Every monoid can be found this way: let $M$ be a monoid. Define $\mathbf{M}$ by saying that $\mathrm{ob}(\mathbf{M}) = \{A\}$, and $\hom(A, A) = M$, where composition of morphisms agree with the binary operation in $M$ and the identity morphism is $1 \in M$. Then the validity of the axioms is clear.

7. Let $S$ be a set with a preorder [i.e. reflexive and transitive] relation $\leq$. Define a category $\mathbf{S}$ by $\mathrm{ob}(\mathbf{S}) = S$, and when $a, b \in S$, $\hom(a, b)$ has exactly one morphism if $a \leq b$, otherwise $\hom(a, b) = \emptyset$. Then $\mathbf{S}$ is a category with composition and identity maps unique determined, and is said to be a **category given by a preorder**.

8. [New categories from old] Let $\mathbf{C}$ and $\mathbf{D}$ be arbitrary categories. Define a new category $\mathbf{C} \times \mathbf{D}$ by specifying that $\mathrm{ob}(\mathbf{C} \times \mathbf{D})$ is the class of pairs $(A, B)$ with $A \in \mathrm{ob}(\mathbf{C})$ and $B \in \mathrm{ob}(\mathbf{D})$, and $\hom_{\mathbf{C}\times\mathbf{D}}((A, B), (C, D)) = \hom_{\mathbf{C}}(A, C) \times \hom_{\mathbf{D}}(B, D)$, the set of pairs of the form $(f, g)$ where $f : A \to C$ and $g : B \to D$. Define $(g, g_1)(f, f_1) = (gf, g_1 f_1)$ when possible and $1_{(A,B)} = (1_A, 1_B)$. This is easily seen to be a category. It is called the **product category of C and D**.

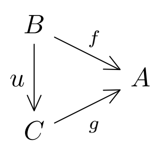
9. If $\mathbf{C}$ is an arbitrary category, the objects in $\mathbf{C}^{\to}$ are the morphisms in $\mathbf{C}$, and whenever $f : A \to B$ and $g : C \to D$ in $\mathbf{C}$, $\hom(f, g)$ is the set of pairs $(h, k)$ with $h : A \to C$ and $k : B \to D$ such that the diagram




is commutative. The hom sets may not be disjoint, but as I said, you can make them disjoint with the use of tags. If $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$ when possible and $1_{(A,B)} = (1_A, 1_B)$, this is also a category, but a bit more interesting.

10. Let $\mathbf{C}$ be an arbitrary category, and define $\mathbf{C}^{\mathrm{op}}$ as follows: $\mathrm{ob}(\mathbf{C}^{\mathrm{op}}) = \mathrm{ob}(\mathbf{C})$; whenever $A, B \in \mathrm{ob}(\mathbf{C})$, $\hom_{\mathbf{C}^{\mathrm{op}}}(A, B) = \hom_{\mathbf{C}}(B, A)$; if $f : A \to B$ and $g : B \to C$ in $\mathbf{C}^{\mathrm{op}}$, define $gf : A \to C$ to be $fg$ as given by $\mathbf{C}$, and $1_A$ in $\mathbf{C}^{\mathrm{op}}$ the same as that in $\mathbf{C}$. This is clearly a category; it is called the **dual category** of $\mathbf{C}$.

11. Let $A$ be an arbitrary object of a category $\mathbf{C}$. Define $\mathbf{C}/A$ as follows: $\mathrm{ob}(\mathbf{C}/A)$ is the class of pairs of the form $(B, f)$ where $B \in \mathrm{ob}(\mathbf{C})$ and $f : B \to A$. For $(B, f), (C, g) \in \mathrm{ob}(\mathbf{C}/A)$, $\hom((B, f), (C, g))$ is the set of morphisms $u : B \to C$ such that $f = gu$, that is,




is commutative. Tag the morphisms to make the hom sets disjoint. Then it is easy to see that $\mathbf{C}/A$ becomes a category by defining composition of morphisms and identity morphisms to agree with $\mathbf{C}$. $\mathbf{C}/A$ is called the **category of objects in C below** $A$.

12. Likewise, define $\mathbf{C}\backslash A$ by agreeing that $\mathrm{ob}(\mathbf{C}\backslash A)$ is the class of pairs of the form $(B, f)$ where $f : A \to B$, and $\hom((B, f), (C, g))$ is the set of morphisms $u : B \to C$ such that $uf = g$. $\mathbf{C}\backslash A$ is then a category, called **the category of objects in C above** $A$.

Note that $\mathrm{ob}(\mathbf{C})$ is a *class*. It may not be a set, as we now see.

$\mathcal{S} = \mathrm{ob}(\mathbf{Set})$ is supposed to be the class of all sets. If $\mathcal{S}$ were a set, we could legally form $X = \{x \in \mathcal{S} \mid x \notin x\}$ [since every $x \in \mathcal{S}$ is a set, $a \in x$ is a defined statement]. But then $X \in X$ if and only if $X \notin X$, so this is an impossible situation. Hence, $\mathcal{S}$ cannot be a set without resulting in paradoxes. So $\mathcal{S}$ is said to be a *proper class*.

In rare occasions, such as in examples 6 and 7 above, **ob(C)** *is* a set. If **C** is a category such that ob(**C**) is a set, **C** is said to be a **small category**.

An **isomorphism** in a category is exactly what one would expect: $f : A \to B$ is an isomorphism provided that there exists $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$. In that case, $g$ is unique, and is denoted $f^{-1}$.

## Subcategories

Remarkably, it already follows that isomorphisms compose into isomorphisms. If $f : A \to B$ is an isomorphism with inverse $f^{-1}$ and $g : B \to C$ is an isomorphism with inverse $g^{-1}$, then $gf : A \to C$ is an isomorphism with inverse $f^{-1}g^{-1} : C \to A$. And of course, $1_A$ is an isomorphism. This illustrates the following definition:

**DEFINITION**

*A category* $\mathbf{D}$ *is said to be a* ***subcategory*** *of a category* $\mathbf{C}$ *provided that:*

(1) *ob*($\mathbf{D}$) *is a subclass of ob*($\mathbf{C}$).

(2) *Whenever* $A, B \in ob(\mathbf{D})$, $\hom_{\mathbf{D}}(A, B)$ *is a subset of* $\hom_{\mathbf{C}}(A, B)$.

(3) *Whenever* $f : A \to B$ *and* $g : B \to C$ *in* $\mathbf{D}$, *the composite* $gf : A \to C$ *given by* $\mathbf{C}$ *is in* $\hom_{\mathbf{D}}(A, C)$ *and is the composite* $gf$ *given by* $\mathbf{D}$.

(4) *For each* $A \in \mathbf{D}$, $\mathbf{C}$*'s identity morphism* $1_A$ *is in* $\hom_{\mathbf{D}}(A, A)$ *as* $\mathbf{D}$*'s identity morphism.*

*If also* $\hom_{\mathbf{D}}(A, B) = \hom_{\mathbf{C}}(A, B)$ *for all* $A, B \in ob(\mathbf{D})$, $\mathbf{D}$ *is said to be a* ***full subcategory*** *of* $\mathbf{C}$.

Notice that subcategories of **C** can be identified purely in terms of their objects and morphisms, because **C** already gives the rest of the structure. And full subcategories can be identified from just the objects! They hypothetically leave all morphisms that they can.

**EXAMPLES**

1. Since every group is a monoid, no two groups can be the same monoid and every group homomorphism is a homomorphism of the monoids, **Grp** is a subcategory of **Mon**. However, **Mon** is *not* a subcategory of **Set**, because a set can be many different monoids. **Mon** *is* a subcategory of **Semgrp** [the semigroups] because a semigroup can't have more than one identity element.

2. Every monoid homomorphism of groups is automatically a group homomorphism, so **Grp** is a full subcategory of **Mon**. However, there exist maps of monoids preserving multiplication which don't map 1 to 1, hence **Mon** is *not* a full subcategory of **Semgrp**.

3. Since a rng can be at most one ring with the same addition and multiplication, **Ring** is a subcategory of **Rng**.

4. Since the isomorphisms in a category **C** are closed under defined composition and involve all identity morphisms, one can form a subcategory of **C** by keeping precisely the isomorphisms.

**EXERCISES**

1. If $F$ and $G$ are fields and $f : F \to G$ is a homomorphism, then $f$ is injective. [*Hint*: What are the ideals in a field?]

2. (a) Suppose $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ are varieties, where every operator for $\mathcal{V}(S_2)$ is in $\mathcal{V}(S_1)$ and every identity in $S_2$ is in $S_1$. Then every $\mathcal{V}(S_1)$ algebra is also a $\mathcal{V}(S_2)$ algebra, and homomorphisms between $\mathcal{V}(S_1)$ algebras are also homomorphisms with $\mathcal{V}(S_2)$'s structure. Assume no two $\mathcal{V}(S_1)$ algebras can have the same $\mathcal{V}(S_2)$ structure. Show that $\mathbf{V}(S_1)$ is a subcategory of $\mathbf{V}(S_2)$.

   (b) If $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ have exactly the same operators, then $\mathbf{V}(S_1)$ is a full subcategory of $\mathbf{V}(S_2)$.

   (c) Show by example that $\mathcal{V}(S_1)$ may have operators that $\mathcal{V}(S_2)$ doesn't, but $\mathbf{V}(S_1)$ is still a full subcategory of $\mathbf{V}(S_2)$.

3. (a) An object $I$ in a category **C** is said to be **initial** provided that for every object $A$, there is exactly one morphism in $\hom(I, A)$. For example, $I_S(\Omega)$ is initial in $\mathbf{V}(S)$ [see Exercise 10 of Section 1.9] and the King variety is initial in **Var** [see Exercise 4 of Section 1.11]. Show that any two initial objects in a category are isomorphic.

   (b) An object $T$ is said to be **terminal** provided that for every object $A$, there is exactly one morphism in $\hom(A, T)$. For example, $T(\Omega)$ is terminal in $\mathbf{V}(S)$, and the variety of sets is terminal in **Var**. Explain why any two terminal objects in a category are isomorphic.

   (c) Whenever $A$ is an object in a category **C**, $\mathbf{C}/A$ has a terminal object, and $\mathbf{C}\backslash A$ has an initial object. [*Hint*: Try $(A, 1_A)$.]

   (d) A **zero object** in a category **C** is an object which is both initial and terminal. If **C** has a zero object, show that one can assign each pair $(A, B)$ of objects in **C** a morphism $0_{A,B} \in \hom(A, B)$ such that $0_{D,B}g = 0_{A,B} = f0_{A,C}$ when they are defined. In particular, show $\hom(A, B)$ is never empty for any objects $A, B \in \mathrm{ob}(\mathbf{C})$.

4. (a) **C** is a full subcategory for **C**, for every category **C**.

   (b) If **E** is a subcategory of **D** and **D** is a subcategory of **C**, then **E** is a subcategory of **C**.

   (c) Prove part (b) with "subcategory" replaced with "full subcategory."

5. A category **C** is discrete if and only if every subcategory of **C** is a full subcategory.

6. Is **Ring** is a full subcategory of **Rng**?

7. (a) Let $\mathcal{V}(S)$ be a variety, and form a category $\mathbf{V}(S)-\mathbf{sub}$ as follows: the objects are the pairs of the form $(A, A_1)$ with $A \in \mathcal{V}(S)$, $A_1$ a subalgebra of $A$, and $\hom((A, A_1), (B, B_1))$ consists of homomorphisms $f : A \to B$ such that $f(A_1) \subseteq B_1$. Verify that this data forms a category with the usual composition of morphisms and identity morphisms.

   (b) Likewise, define $\mathbf{V}(S)-\mathbf{con}$ as follows: the objects are the pairs of the form $(A, \Phi)$ with $A \in \mathcal{V}(S)$, $\Phi$ a congruence relation on $A$, and $\hom((A, \Phi), (B, \Theta))$ consists of homomorphisms $f : A \to B$ such that whenever $a\Phi b$ in $A$, $f(a)\Theta f(b)$ in $B$. Then $\mathbf{V}(S)-\mathbf{con}$ is a category.

8. A small category $\mathbf{C}$ in which all morphisms are isomorphisms is called a **groupoid**. In this exercise we establish a non-categorial definition of a groupoid. We see it as a set $G$ equipped with the following structure:

   (1) Whenever $a, b \in G$, $ab$ is either some element of $G$ or is *undefined*. [This is a **partial operator**.]

   (2) Whenever $a \in G$, $a^{-1}$ is some element of $G$.

   (3) [associativity] Whenever $ab$ and $bc$ are defined in $G$, then $(ab)c$ and $a(bc)$ are defined and $(ab)c = a(bc)$.

   (4) [inverse] $aa^{-1}$ and $a^{-1}a$ are always defined for $a \in G$.

   (5) [identity] Whenever $ab$ is defined in $G$, $abb^{-1} = a$ and $a^{-1}ab = b$. [Note that rules (3) and (4) already show that those expressions are defined and unambiguous.]

   (a) If $\mathbf{C}$ is a groupoid, consider $G = \biguplus_{A,B\in \mathrm{ob}(\mathbf{C})} \hom(A, B)$. If $a : A \to B$ and $b : A' \to B'$ are in $G$, define $ab$ to be $ab : A' \to B$ as given in $\mathbf{C}$ if $A = B'$, and undefined if $A \neq B'$. Then define $a^{-1}$ to be $a^{-1} : B \to A$ as given in $\mathbf{C}$. Verify rules (3), (4), (5) for $G$.

   Now suppose $G$ is any set equipped with structure satisfying the five rules above. Show that for $a, b \in G$:

   (b) $(a^{-1})^{-1} = a$. [*Hint*: Why is $(a^{-1})^{-1}a^{-1}a$ defined? Change it in two ways.]

   (c) If $ab$ is defined, then $b^{-1}a^{-1}$ is defined and $b^{-1}a^{-1} = (ab)^{-1}$. [*Hint*: $b^{-1}a^{-1}ab$ and $ab(ab)^{-1}$ are defined [why?].]

   (d) For $a, b \in G$, define $a\Phi b$ to mean that $ab^{-1}$ is defined. Then $\Phi$ is an equivalence relation on $G$.

   (e) If $a \in G$, $T(a) = \{x \in G \mid xa \text{ is defined}\}$ is an equivalence class of $\Phi$, which may be different from $a$'s. [Yes, this means it must be nonempty.]

   (f) Define $\mathrm{ob}(\mathbf{C}) = G/\Phi$, the set of equivalence classes, and for $A, B \in G/\Phi$, $\hom(A, B) = \{a \in A \mid T(a) = B\}$. If $a \in \hom(A, B)$ and $b \in \hom(B, C)$, $ba \in \hom(A, C)$. Also, $a^{-1}a$ is the same for all $a \in A$, and is an identity morphism in $\hom(A, A)$. Conclude that $\mathbf{C}$ is a groupoid in the categorical sense.

(g) Take a look at the translations for a groupoid in (a) and (f). If you go through one of them and then the other, must you end up with the same thing you started with?

# 2.2 - Monic and Epic Morphisms

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(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

The injectivity and surjectivity of $\mathcal{V}(S)$-algebras was quite important. This importance carries over to category theory. Unfortunately, for morphisms which don't go from sets to sets, injectivity and surjectivity can hardly be defined. Let's explore the $\mathcal{V}(S)$-algebras, once again.

Suppose $f : A \to B$ is an injective homomorphism in $\mathbf{V}(S)$. If $C \in \mathcal{V}(S)$, $g, h : C \to A$ and $fg = fh$, then for all $a \in A$, $fg(a) = fh(a)$. Thus $f(g(a)) = f(h(a))$, which implies $g(a) = h(a)$ since $f$ is injective. Therefore, $g = h$.

But what if $f$ isn't injective? Let $C = \ker f = \{(a, b) \in A \times A \mid f(a) = f(b)\}$. Then $C \in \mathcal{V}(S)$. Define $g, h : C \to A$ by $g(a, b) = a$ and $h(a, b) = b$. For all $(a, b) \in C$, $f(a) = f(b)$ by definition, so that $fg(a, b) = fh(a, b)$. Therefore, $fg = fh$. However, since $f$ is not injective, there exist $a \neq b$ in $A$ such that $f(a) = f(b)$. Furthermore, $(a, b) \in C$ and $g(a, b) \neq h(a, b)$, so that $g \neq h$.

We have shown

**A homomorphism $f : A \to B$ in $\mathbf{V}(S)$ is injective if and only if $fg = fh$ always implies $g = h$ for homomorphisms $g, h : C \to A$.**

This defines injectivity of a homomorphism using purely maps, leading to the following definition and theorem.

**DEFINITION**

*If $\mathbf{C}$ is a category, and $f : A \to B$ in $\mathbf{C}$, $f$ is said to be **monic** provided that whenever $fg = fh$ in $\mathbf{C}$, $g = h$.*

**THEOREM 2.1a** *A homomorphism in $\mathbf{V}(S)$ is monic if and only if it's injective.*

How how would one define surjectivity? There is a bit of less luck here. Suppose $f : A \to B$ is surjective in $\mathbf{V}(S)$. If $C \in \mathcal{V}(S)$, $g, h : B \to C$ and $gf = hf$, then for all $b \in B$, $b = f(a)$ for some $a \in A$ since $f$ is surjective, and $g(b) = gf(a) = hf(a) = h(b)$. Hence, $g = h$.

But in rare cases, we could still have $gf = hf \implies g = h$ when $f$ is not surjective. Consider the canonical monomorphism of rings $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$. If $g : \mathbb{Q} \to R$ and $h : \mathbb{Q} \to R$ are ring homomorphisms such that $gf = hf$, then $g|\mathbb{Z} = h|\mathbb{Z}$. Exercise 1 shows that $g = h$ follows. So this property of maps *strictly contains* surjectivity:

**DEFINITION**

*If $\mathbf{C}$ is a category, and $f : A \to B$ in $\mathbf{C}$, $f$ is said to be **epic** provided that whenever $gf = hf$ in $\mathbf{C}$, $g = h$.*

The preceding example shows that the canonical monomorphism $\iota : \mathbb{Z} \to \mathbb{Q}$ in **Ring** is epic but not surjective, so we have:

**THEOREM 2.1b** *A surjective homomorphism in* $\mathbf{V}(S)$ *is epic, but not conversely.*

A complete classification of epic homomorphisms in $\mathbf{V}(S)$ is found in Exercise 2.

Theorem 2.1a may be false when the category consists of $\Omega$-algebras but isn't a variety. To see an example, define a **divisible abelian group** to be an abelian group $G$ [written additively] such that for all $a \in G$, $0 \neq n \in \mathbb{Z}$, there exists $b \in G$ with $a = nb$. Thus you can divide by any nonzero integer in $G$.

Let $\mathbf{Ab} - \mathbf{div}$ be the full subcategory of $\mathbf{Ab}$ consisting of the divisible abelian groups. Clearly $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible; let $f : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the canonical epimorphism. We claim that $f$ is monic, even though it is not injective: suppose $G$ is a divisible abelian group and $g, h : G \to \mathbb{Q}$ are homomorphisms with $fg = fh$. Then $g - h : G \to \mathbb{Q}$ is a homomorphism and $f(g - h) = 0$, implying that $\operatorname{im}(g - h) \subseteq \ker f = \mathbb{Z}$. Evidently a homomorphic image of a divisible abelian group is divisible, so $\operatorname{im}(g - h)$ is divisible [since $G$ is]. The only subgroup of $\mathbb{Z}$ which is divisible is 0, and hence, $g - h$ is the zero map and $g = h$. Therefore, $f$ is monic.

The conclusion is that if categories consist of sets and maps, monicness and epicness are only *approximations* of injectivity and surjectivity.

Now for a few basic properties about morphisms in an arbitrary category. For example, we know that if $f : A \to B$ and $g : B \to C$ are both injective in **Set**, $gf$ is injective. The same is true for monic morphisms, as we now see.

**THEOREM 2.2** *Let* $\mathbf{C}$ *be a category and* $f : A \to B$, $g : B \to C$ *morphisms in* $\mathbf{C}$.

(1) *If* $f$ *and* $g$ *are monic,* $gf$ *is monic.*
(2) *If* $gf$ *is monic, then* $f$ *is monic.*
(3) *If* $f$ *and* $g$ *are epic,* $gf$ *is epic.*
(4) *If* $gf$ *is epic, then* $g$ *is epic.*

*Proof of Theorem 2.2.* (1) If $gfx = gfy$ with $x, y : D \to A$, then $fx = fy$ since $g$ is monic, hence $x = y$ since $f$ is monic.

(2) If $fx = fy$ with $y : D \to A$, then $gfx = gfy$, so that $x = y$ since $gf$ is monic.

(3) and (4) have essentially the same proof with the arrows reversed. ■

We now define subobjects and quotient objects of an arbitrary object in a category.

Fix $A \in \operatorname{ob}(\mathbf{C})$. Consider the class of all monic morphisms from any object in $\mathbf{C}$ to $A$. If $f$ and $g$ are such morphisms, define $f \leq g$ provided that $f = gk$ for some $k$ [which is theoretically monic]. Then evidentiy $\leq$ is reflexive and transitive. Now define $f \equiv g$ provided that $f \leq g$ and $g \leq f$; equivalently,

$f = gk$ for some *isomorphism* $k$. $\equiv$ is an equivalence relation, and its equivalence classes are the **subobjects** of $A$.

Furthermore, if $f \equiv f'$ and $g \equiv g'$, then $f \leq g$ if and only if $f' \leq g'$. Thus $\leq$ is actually a partial order on the subobjects of $A$.

The special case where $\mathbf{C} = \mathbf{V}(S)$ considers all injective homomorphisms into $A$. If $f$ and $g$ are such homomorphisms, $f \leq g$ if and only if $\operatorname{im} f \subseteq \operatorname{im} g$. Thus, $f \equiv g$ if and only if they have the same image. Since the possible images of injective homomorphisms into $A$ are the subalgebras of $A$, the above definition makes sense.

Now consider the class of all epic morphisms from $A$ to any object in $\mathbf{C}$. If $f$ and $g$ are such morphisms, define $f \geq g$ to mean $f = kg$ for some $k$, and $f \equiv g$ to mean $f \geq g$ and $g \geq f$. Once again, $\equiv$ is an equivalence relation, its equivalence classes are the **quotient objects** of $A$, and $\geq$ is a partial order on them.

If $f$ and $g$ are surjective homomorphisms from $A$ in $\mathbf{V}(S)$, then $f \geq g$ if and only if $\ker f \supseteq \ker g$, and $f \equiv g$ if and only if they have the same kernel. Since the possible kernels are the congruence relations on $A$, the above definition would make sense, except for the trap that epic homomorphisms need not be surjective. Because of this, the class of quotient objects of $A$ as defined above may be larger than the actual class of quotient algebras.

**CAUTION** In universal algebra, people refer to injective homomorphisms as **monomorphisms**, and surjective homomorphisms as **epimorphisms**. In category theory, however, a "monomorphism" refers to a monic morphism and an "epimorphism" refers to an epic morphism. You need to watch out which definitions are being used when, because they are not the same!

**EXERCISES**

1. Suppose $g : \mathbb{Q} \to R$ and $h : \mathbb{Q} \to R$ are ring homomorphisms with $g|\mathbb{Z} = h|\mathbb{Z}$.

   (a) For each $n \neq 0$ in $\mathbb{Z}$, $g(n)$ is a two-sided unit in $R$. [*Hint*: Multiply by $g(1/n)$ on both sides.]

   (b) For each $n \neq 0$ in $\mathbb{Z}$, $g(1/n) = h(1/n)$.

   (c) $g = h$. Hence, the canonical monomorphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$, though not surjective, is epic.

2. (UNIVERSAL ALGEBRA) If $A$ and $B$ are $\mathcal{V}(S)$-algebras and $g, h : A \to B$ are homomorphisms, let $K = \{a \in A \mid g(a) = h(a)\}$. We already know that $K$ is a subalgebra of $A$ [Exercise 10(a) of Section 1.3]. $K$ is called the **difference kernel** of $g$ and $h$.

   (a) If $f : C \to A$ is a homomorphism, then $gf = hf$ if and only if $f(C) \subseteq K$.

   (b) If $B$ is a subalgebra of $A$, then $B$ is said to be **nice** provided that there exists an $\mathcal{V}(S)$-algebra $C$ and homomorphisms $g, h : A \to C$ with difference kernel $B$. Show that $A$ is nice in $A$, and the intersection of any batch of nice subalgebras of $A$ is nice. [*Hint*: Products.] Conclude that Theorem 1.3 can be applied to define the nice subalgebra generated by a set.

(c) A homomorphism $f : A \to B$ is epic if and only if $B$ is the *nice* subalgebra of $B$ generated by $f(A)$.

(d) Every submodule of an $R$-module is nice. Conclude that epic morphisms are surjective in $R-$**mod**.

(e) If $R$ is a ring and $S$ is a nice subring of $R$, then whenever $u \in S$ is a unit in $R$, then $u^{-1} \in S$. Conclude that $\mathbb{Z}$ is *not* a nice subring of $\mathbb{Q}$.

3. (UNIVERSAL ALGEBRA) If $A$ and $B$ are $\mathcal{V}(S)$-algebras and $g, h : A \to B$ are homomorphisms, let $\Theta$ be the congruence relation on $B$ generated by $\{(f(a), g(a)) \mid a \in A\}$. $\Theta$ is called the **difference image** of $f$ and $g$.

   (a) If $f : B \to C$ is a homomorphism, then $fg = fh$ if and only if $\Theta \subseteq \ker f$.

   (b) For *every* congruence relation $\Phi$ on $B$, there exists an $\mathcal{V}(S)$-algebra $A$ and homomorphisms $g, h : A \to B$ with difference image $\Phi$. Explain why this implies that monics are injective in $\mathbf{V}(S)$.

4. In **Grp**, epic morphisms are surjective. [*Hint*: Suppose $f : G \to H$ is a group homomorphism which is not surjective. If $I = f(G)$ is normal in $H$, the proof that $f$ is not epic should be easy. Otherwise, $[H : I] \geq 3$ [why?] Let $S(H)$ be the group of permutations of the *set* $H$ and define $g : H \to S(H)$ by sending every $a \in H$ to $a_L \in S(H)$ sending $x \to ax$. Show that there exists $p \in S(H)$ which commutes with every $a_L$ with $a \in I$, but fails to commute with some $a_L$ with $a \in H - I$. Then, if $h : H \to S(H)$ is defined by $h(a) = pa_Lp^{-1}$, show that $gf = hf$ but $g \neq h$.]

5. If $f : A \to B$ and $g : B \to A$ are morphisms in $\mathbf{C}$ such that $gf = 1_A$, $f$ is said to be a **section** of $g$ and $g$ is said to be a **retraction** of $f$. Hence, a morphism is a section if and only if it has a retraction, and a morphism is a retraction if and only if it has a section.

   (a) Sections are monic and retractions are epic in $\mathbf{C}$, but not conversely.

   (b) Show by example that a morphism in $\mathbf{C}$ may have more than one section, or more than one retraction.

   (c) If $f$ has both a section and a retraction, then $f$ is an isomorphism, and the conditions in (b) can't hold.

   (d) Comment on how this links to universal algebra. [*Hint*: See Exercise 10 of Section 1.5]

# 2.3 - Functors and Natural Transformations

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

Functors are structure-preserving maps of categories. However, since categories are more than just classes of objects, it appears that we haven't really seen a map of categories before. However, we have; consider takeoffs from Section 1.11, but this time regard $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ as categories.

A takeoff gives every $\mathcal{V}(S_1)$-algebra a $\mathcal{V}(S_2)$-algebra structure in such a way that all $\mathcal{V}(S_1)$-homomorphisms preserve the $\mathcal{V}(S_2)$-structure. One can think about this as assigning a $\mathcal{V}(S_1)$-algebra a $\mathcal{V}(S_2)$-algebra and every homomorphism of $\mathcal{V}(S_1)$-algebras a homomorphism of the corresponding $\mathcal{V}(S_2)$-algebras. The composition of maps and identity maps are obviously preserved by this.

Abstracting the above information, we have the following definition:

**DEFINITION**

*If* $\mathbf{C}$ *and* $\mathbf{D}$ *are categories, a* ***[covariant] functor*** *from* $\mathbf{C}$ *to* $\mathbf{D}$ *is a mathematical object* $F$ *such that:*

(1) *For each* $A \in ob(\mathbf{C})$, $FA$ *is some object of* $\mathbf{D}$.

(2) *For each* $f : A \to B$ *in* $\mathbf{C}$, $F(f)$ *is some morphism* $FA \to FB$ *of* $\mathbf{D}$.

(3) $F(gf) = F(g)F(f)$ *whenever* $gf$ *is defined in* $\mathbf{C}$.

(4) $F(1_A) = 1_{FA}$ *for all* $A \in ob(\mathbf{C})$.

**EXAMPLES**

1. A takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ of varieties becomes a functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$ sending every $\mathcal{V}(S_1)$-algebra to itself as a $\mathcal{V}(S_2)$-algebra with the derived structure, and every homomorphism of $\mathcal{V}(S_1)$-algebras to itself. In particular, for any variety $\mathcal{V}(S)$, we have the **forgetful functor** $F : \mathbf{V}(S) \to \mathbf{Set}$ sending every algebra to its underlying set, and every homomorphism to itself as a set map.

2. If $\mathbf{M}$ and $\mathbf{N}$ are monoids viewed as one-object categories [Example 6 of Section 1], a functor $\mathbf{M} \to \mathbf{N}$ is a monoid homomorphism. If $\mathbf{S}$ and $\mathbf{T}$ are preordered sets viewed as categories with at most one morphism in every hom set [Example 7 of Section 1], a functor $\mathbf{S} \to \mathbf{T}$ is an order-preserving map.

3. Let **Poset** be the category consisting of partially ordered sets and order-preserving maps. Evidently a homomorphism $f : A \to B$ of $\mathcal{V}(S)$-algebras induces an order-preserving map from Sub $A$ to Sub $B$. Hence we have a functor $\text{Sub} : \mathbf{V}(S) \to \mathbf{Poset}$ sending every $\mathcal{V}(S)$-algebra to its subalgebra lattice.

4. If $\mathbf{D}$ is a subcategory of $\mathbf{C}$, one can form the **injection functor** $I : \mathbf{D} \to \mathbf{C}$ sending every object and morphism in $\mathbf{D}$ to itself in $\mathbf{C}$. Thus $IA = A$ and $I(f) = f$ for $A \in \text{ob}(\mathbf{D}), f \in \text{hom}_{\mathbf{D}}(A, B)$. The special case when $\mathbf{D} = \mathbf{C}$ is the **identity functor** $1_{\mathbf{C}}$.

5. Fix an object $B \in \mathbf{D}$, then we have the **constant functor** $F : \mathbf{C} \to \mathbf{D}$ defined by $FA = B$ for all $A \in \text{ob}(\mathbf{C})$ and $F(f) = 1_B$ for all $f : A \to A'$ in $\mathbf{C}$.

6. If $\mathbf{C}$ is a discrete category, a functor $\mathbf{C} \to \mathbf{D}$ simply assigns each object of $\mathbf{C}$ an object of $\mathbf{D}$.

7. Define a functor $U : \mathbf{Ring} \to \mathbf{Grp}$ as follows: For each ring $R$, $UR$ is its group of units. Since every ring homomorphism $f : R \to S$ sends units to units, it restricts to a group homomorphism $U(f) : UR \to US$.

8. For each ring $R$, one can form the polynomial ring $R[x]$ by adjoining a symbol $x$ and agreeing that $xr = rx$ for all $r \in R$. Then every element of $R[x]$ is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with the $a_i$'s in $R$. Evidently a homomorphism $f : R \to S$ induces one $\overline{f} : R[x] \to S[x]$ by defining

$$\overline{f}(a_n x^n + \cdots + a_1 x + a_0) = f(a_n)x^n + \cdots + f(a_1)x + f(a_0)$$

This is a functor $\mathbf{Ring} \to \mathbf{Ring}$, sending every ring $R$ to the polynomial ring $R[x]$ and every homomorphism $f$ to $\overline{f}$.

9. Another functor $M_n : \mathbf{Ring} \to \mathbf{Ring}$ sends each ring $R$ to the matrix ring $M_n(R)$ and each homomorphism $f : R \to S$ to the homomorphism $\tilde{f} : M_n(R) \to M_n(S)$ defined by

$$\tilde{f}(\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix}) = \begin{bmatrix} f(r_{11}) & f(r_{12}) & \dots & f(r_{1n}) \\ f(r_{21}) & f(r_{22}) & \dots & f(r_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ f(r_{n1}) & f(r_{n2}) & \dots & f(r_{nn}) \end{bmatrix}$$

10. Let $\mathbf{M}$ be a monoid $M$, regarded as a category with one object [Example 6 of Section 1]. Then a functor $\mathbf{M} \to \mathbf{Set}$ is an $M$-action. This is because $\mathbf{M}$ has only one object, so it is assigned by the functor to only one set. In fact, a functor $\mathbf{M} \to \mathbf{V}(S)$ is a $\mathcal{V}(S)$-representation of $M$.

11. Let $R$ be a fixed ring, $n$ a fixed positive integer. Whenever $M$ is an $R$-module, $M^n$ becomes a left module over the matrix ring $M_n(R)$ when defined as follows.

$$\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} r_{11}a_1 + r_{12}a_2 + \cdots + r_{1n}a_n \\ r_{21}a_1 + r_{22}a_2 + \cdots + r_{2n}a_n \\ \vdots \\ r_{n1}a_1 + r_{n2}a_2 + \cdots + r_{nn}a_n \end{bmatrix}$$

And for every $R$-module homomorphism $f : M \to N$, the map $f^n : M^n \to N^n$ sending $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \to \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_n) \end{bmatrix}$ is an $M_n(R)$-module homomorphism. Consequently, this defines a functor $R\mathbf{-mod} \to M_n(R)\mathbf{-mod}$.

12. Example 1 shows that the takeoff $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ of varieties becomes a functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$. We proceed to construct a functor the other way $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$. For each $A \in \mathbf{V}(S_2)$, there is a unique [up to isomorphism] universal $\mathcal{V}(S_1)$-algebra $(U_A, i_A)$ enveloping $B$ for the takeoff, by Theorems 1.28 and 1.29. Define $GA = U_A$. For $f : A \to B$ in $G$, to define $G(f) : U_A \to U_B$, note that $i_B f$ is an $\Omega_2$-homomorphism $A \to U_B$. Therefore there exists a unique

$\Omega_1$-homomorphism $h : U_A \to U_B$ such that $i_B f = h i_A$; define $G(f) = h$. Then $G(f)i_A = i_B f$ for all $f : A \to B$ in $\mathbf{V}(S_2)$, that is,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ U_A & \xrightarrow{G(f)} & U_B \end{array}$$

is commutative.

Now suppose $f : A \to B$ and $g : B \to C$ in $\mathbf{V}(S_2)$. Then $G(gf)$ is the *unique* morphism $GA \to GC$ in $\mathbf{C}$ satisfying $G(gf)i_A = i_C gf$. However, $G(g)G(f)$ also satisfies this statement, because the commutativity of the squares in

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} & & \downarrow{\scriptstyle i_C} \\ U_A & \xrightarrow{G(f)} & U_B & \xrightarrow{G(g)} & U_C \end{array}$$

implies that

$$\begin{array}{ccc} A & \xrightarrow{gf} & C \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_C} \\ U_A & \xrightarrow{G(g)G(f)} & U_B \end{array}$$

is commutative. Therefore, $G(gf) = G(g)G(f)$ by uniqueness. Likewise, for $A \in \mathcal{V}(S_2)$, $G(1_A)$ is the unique morphism $U_A \to U_A$ satisfying $G(1_A)i_A = i_A 1_A$. But obviously $1_{GA}$ satisfies that statement; whence $G(1_A) = 1_{GA}$. Therefore, $G$ is a functor. It is a *left adjoint functor* of $F$, and that will be studied in Section 8.

The special case where $\mathcal{V}(S_2)$ is the variety of sets induces the **free-algebra functor** $G : \mathbf{Set} \to \mathbf{V}(S)$, sending every set $X$ to $F_S(\Omega, X)$, and every set map $f : X \to Y$ the unique homomorphism $F_S(\Omega, X) \to F_S(\Omega, Y)$ which extends $i_Y f : X \to F_S(\Omega, Y)$.

13. A **functor in two variables** refers to a functor from a product category. For example, if $\mathbf{C}$ and $\mathbf{D}$ are categories, one cn form the **projection functor** $P_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}$ by $P_1(A, B) = A$, $P_1(f, g) = f$. The other projection $P_2$ is defined similarly.

14. The **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{C}\times\mathbf{C}$ sends $A \to (A, A)$ and $f \to (f, f)$ for $f : A \to A'$ in $\mathbf{C}$.

A functor $F$ is said to be **faithful** provided for all $A, B \in \mathrm{ob}(\mathbf{C})$, the map $\hom(A, B) \to \hom(FA, FB)$ sending $f \to F(f)$ is injective. In other words, $F(f) = F(g)$ for $f, g : A \to B$ imply $f = g$. But be careful; this *doesn't* mean $F$ is an injective map on the objects. For example, the the functor in Example 1 is faithful, but not necessarily injective on the objects. Example 3 is both faithful and injective on the objects. Example 13 is not faithful though.

A functor $F$ is said to be **full** provided that for all $A, B \in \text{ob}(\mathbf{C})$, the map $\hom(A, B) \to \hom(FA, FB)$ sending $f \to F(f)$ is surjective. In other words, every morphism $FA \to FB$ is of the form $F(f)$ with $f : A \to B$. Take caution again: this doesn't mean $F$ is a surjective map on the objects. The functor given by a takeoff [Example 1] is full if and only if the takeoff is full in the sense of Section 1.11, Exercise 8. Also, if $\mathbf{D}$ is a subcategory of $\mathbf{C}$, then the injection functor $\mathbf{D} \to \mathbf{C}$ is full if and only if $\mathbf{D}$ is a full subcategory, which explains the terminology.

If $\mathbf{C}$ and $\mathbf{D}$ are categories, a **contravariant functor** from $\mathbf{C}$ to $\mathbf{D}$ is a functor from $\mathbf{C}^{\text{op}}$ to $\mathbf{D}$. [$\mathbf{C}^{\text{op}}$ is defined in Example 10 of Section 1.] Specifically, it's an object $F$ such that:

(1) For each $A \in \text{ob}(\mathbf{C})$, $FA$ is some object of $\mathbf{D}$.
(2) For each $f : A \to B$ in $\mathbf{C}$, $F(f)$ is some morphism $FB \to FA$ of $\mathbf{D}$.
(3) $F(gf) = F(f)F(g)$ whenever $gf$ is defined in $\mathbf{C}$.
(4) $F(1_A) = 1_{FA}$ for all $A \in \text{ob}(\mathbf{C})$.

Statement (2), for example, is understood in the sense that $f \in \hom_{\mathbf{C}^{\text{op}}}(B, A)$.

There are many kinds of functors which reverse the arrows like these, seen in the following examples. To avoid confusion, the plain word "functor" will always mean "covariant functor".

**EXAMPLES**

1. If all morphisms in $\mathbf{C}$ are isomorphisms, one can form a contravariant functor $\text{Inv} : \mathbf{C} \to \mathbf{C}$ sending every object to itself and every morphism to its inverse. The conditions are readily verified. It is called the **inversion functor**.

2. If $\mathbf{M}$ and $\mathbf{N}$ are monoids viewed as one-object categories, a contravariant functor $\mathbf{M} \to \mathbf{N}$ is a monoid antihomomorphism. If $\mathbf{S}$ and $\mathbf{T}$ are preordered sets viewed as categories with at most one morphism in every hom set, a contravariant functor $\mathbf{S} \to \mathbf{T}$ is an order-reversing map.

3. Define a contravariant functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ as follows: For each $A \in \text{ob}(\mathbf{Set})$, $\mathcal{P}A$ is the power set $\mathcal{P}(A)$, and for each set map $f : A \to B$, $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ sends every $X \subseteq B$ to its preimage $f^{-1}(X)$. Elementary set theory shows that $(gf)^{-1}(X) = f^{-1}(g^{-1}(X))$ when they are defined and $1_A^{-1}(X) = X$. Therefore, $\mathcal{P}$ is a contravariant functor.

In fact, since the preimage map is a Boolean algebra homomorphism [check this!] one could take the category of Boolean algebras as the codomain of $\mathcal{P}$, instead of **Set**.

4. Let $M$ be a fixed monoid. We define a contravariant functor $D : M{-}\mathbf{act} \to \mathbf{act}{-}M$ as follows. For each left $M$-action $X$, define $X^* = \hom(X, M)$, the set of $M$-action homomorphisms from $X$ to $M$ [with the obvious $M$-action structure]. To make $X^*$ into a right $M$-action, take each $\varphi \in X^*, m \in M$. Define $\varphi m : X \to M$ by $\varphi m(x) = \varphi(x)m$ with $x \in X$. [This uses the monoid multiplication in $M$.] It is then straightforward that $X^*$ is a right $M$-action. It is called the **dual** of the left $M$-action $X$.

Now suppose $f : X \to Y$ is a homomorphism of $M$-actions. Define the **transposed map** $f^* : Y^* \to X^*$ by $f^*(\varphi) = \varphi f$ with $\varphi \in Y^*$. We claim that

$f^*$ is a homomorphism; for $\varphi \in Y^*, m \in M, x \in X$,

$$f^*(\varphi m)(x) = (\varphi m)(f(x)) = \varphi(f(x))m = f^*(\varphi)(x)m = (f^*(\varphi)m)(x)$$

Therefore, $f^*(\varphi m) = f^*(\varphi)m$. If one defines $DX = X^*, D(f) = f^*$, it is clear that $D$ is a contravariant functor. It is called the **dual functor**.

One can likewise define a dual functor from $\mathbf{act}{-}M$ to $M{-}\mathbf{act}$. The details are left to the reader.

5. If $R$ is a fixed ring, the same details from the previous example establish the dual functors $D : R{-}\mathbf{mod} \to \mathbf{mod}{-}R$ and $D : \mathbf{mod}{-}R \to R{-}\mathbf{mod}$: for every $R$-module $M$, let $M^* = \hom(M, R)$, and for every homomorphism $f : M \to N$, define $f^* : N^* \to M^*$ by $f^*(\varphi) = \varphi f$. But this time, one must take addition into account. Then assign $DM = M^*, D(f) = f^*$.

## Do the categories form a category?

If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are functors, one defines the **composite functor** $GF : \mathbf{C} \to \mathbf{E}$ by $(GF)A = G(FA)$ for $A \in \mathrm{ob}(\mathbf{C})$, $(GF)(f) = G(F(f))$ for $f : A \to B$ in $\mathbf{C}$. Evidently this is a functor. Likewise, if $F$ or $G$ is contravariant, then $GF$ can be defined this way: $GF$ is covariant if $F$ and $G$ are both covariant or both contravariant; $GF$ is contravariant if one of $F, G$ is covariant and the other contravariant.

It is clear that $(HG)F = H(GF)$ when the compositions are defined, and when $F : \mathbf{C} \to \mathbf{D}$, $F1_{\mathbf{C}} = F = 1_{\mathbf{D}}F$. And certainly the domain and codomain of a functor are intrinsic. Doesn't this mean that *there's a category whose objects are categories and whose morphisms are covariant functors between categories*? They satisfy everything in the definition, don't they? But isn't it a bit fishy that one of the categories is to be made up of all of them? Well, the question is controversial. Some argue that the answer is "no":

(1) In the definition of a category, we said $\hom(A, B)$ has to be a set. However, the class of functors from $\mathbf{C}$ to $\mathbf{D}$ need not be a set.

(2) Just like a batch of sets doesn't necessarily form a set, a batch of classes — such as categories — might not be able to form a class. What they form would be a "conglomerate" at best.

(3) One would be able to form the full subcategory of categories that don't contain themselves as objects. This category contains itself as an object if and only if it doesn't, causing a paradox.

We shall be bold as to disagree with all those arguments [for example, (3) doesn't work because set-builder notation can't be used on a proper class]. Thus we assume that the category of all categories can be formed, and we call it **Cat**. Meow if you find this cute!

## Natural Transformations

If you're overburdened by the fact that categories [whose role is to hold morphisms] have morphisms of their own [the functors], prepare to know that functors have their own morphisms as well! These morphisms, called natural transformations, actually have the slightest bit of information in their hands. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors, a natural transformation from $F$ to $G$ assigns each object $A$ in $\mathbf{C}$ a morphism $FA \to GA$ such that the morphisms have the "same state of mind" and are therefore compatible with where $F$ and $G$ send morphisms. This is formalized in the definition:

**DEFINITION**

*Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $F, G : \mathbf{C} \to \mathbf{D}$ *functors. A* ***natural transformation*** *from* $F$ *to* $G$ *is a mathematical object* $\eta$ *assigning each* $A \in ob(\mathbf{C})$ *to a morphism* $\eta_A \in \hom_{\mathbf{D}}(FA, GA)$ *such that for all* $f : A \to B$ *in* $\mathbf{C}$*, the diagram*

$$\begin{CD} FA @>{\eta_A}>> GA \\ @V{F(f)}VV @VV{G(f)}V \\ FB @>{\eta_B}>> GB \end{CD}$$

*is commutative. If every* $\eta_A$ *is an isomorphism,* $\eta$ *is called a* ***natural isomorphism****.*

**EXAMPLES**

1. Let $T : \mathbf{Ring} \to \mathbf{Mon}$ be the takeoff from rings to monoids, regarding the multiplication and forgetting the addition. Then, let $U : \mathbf{Ring} \to \mathbf{Mon}$ be the functor of Example 6 — since groups are monoids, we can change the codomain this way! For each ring $R$, $TR$ is the multiplicative monoid of $R$, whereas $UR$ is the group of units of $R$. It turns out that $UR$ is solely the group of units of the monoid $TR$, and one can consider the canonical monomorphism $\eta_R : UR \hookrightarrow TR$ in $\mathbf{Mon}$. It is straightforward to see that if $\eta_R$ is defined that way for every ring $R$, $\eta$ is a natural transformation from $U$ to $T$.

2. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties, and $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$, $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$ be the functors in Examples 1 and 12. For each $A \in \mathcal{V}(S_2)$, $GA$ is the universal $\mathcal{V}(S_1)$ algebra enveloping $A$, let $i_A : A \to GA$ be the $\mathcal{V}(S_2)$ map. Then $i_A$ is really a morphism from $A$ to $FGA$ because it regards the $\mathcal{V}(S_1)$ algebra $GA$ with the derived structure. The statement $G(f)i_A = i_B f$, which is really $FG(f)i_A = i_B f$, says that $A \to i_A$ is a natural transformation from $1_{\mathbf{V}(S_2)}$ to $FG$.

3. Fix a monoid $M$, and recall the dual functors $D : M{-}\mathbf{act} \to \mathbf{act}{-}M$ and $\mathbf{act}{-}M \to M{-}\mathbf{act}$. Composing them yields the **double dual functor** $D^2 : M{-}\mathbf{act} \to M{-}\mathbf{act}$. It sends each $M$-action $X$ to $X^{**} = \hom(\hom(X, M), M)$. For each $X$, define $\eta_X : X \to X^{**}$ by agreeing that $\eta_X(x)$ for each $x \in X$ is the map $\hom(X, M) \to M$ sending $\varphi \to \varphi(x)$. Thus $\eta_X(x)(\varphi) = \varphi(x)$. We claim that $\eta$ is a natural transformation from $1_{M-\mathbf{act}} \Rightarrow D^2$.

First we must show that $\eta_X$ has a good target in the sense that $\eta_X(x)$ is actually a homomorphism of right $M$-actions. To do this, we need to show that $\eta_X(x)(\varphi m) = \eta_X(x)(\varphi)m$. Well, $\eta_X(x)(\varphi m) = \varphi m(x) = \varphi(x)m$ (by definition of $\varphi m) = \eta_X(x)(\varphi)m$.

Next we show that each $\eta_X$ is a homomorphism of left $M$-actions: For all $x \in X, m \in M, \varphi \in X^*$, $\eta_X(mx)(\varphi) = \varphi(mx) = m\varphi(x) = m\eta_X(x)(\varphi)$, since $\varphi \in \hom(X, M)$ is a left $M$-action homomorphism. Therefore, $\eta_X(mx) = m\eta_X(x)$.

Finally we claim that $\eta$ is a natural transformation from $1_{M-\mathbf{act}}$ to $D^2$. Let $f : X \to Y$ be any $M$-action homomorphism. Then $f^* : Y^* \to X^*$ is given by $f^*(\varphi) = \varphi f$, and $f^{**} : X^{**} \to Y^{**}$ is given by $f^{**}(\psi) = \psi f^*$. Thus $f^{**}(\psi)(\varphi) = \psi f^*(\varphi) = \psi(f^*(\varphi)) = \psi(\varphi f)$. To show that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & X^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ Y & \xrightarrow{\eta_Y} & Y^{**} \end{array}$$

is commutative, we work as follows:

$$f^{**}(\eta_X(x))(\varphi) = \eta_X(x)(\varphi f) = \varphi f(x) = \varphi(f(x))$$

$$\eta_Y f(x)(\varphi) = \eta_Y(f(x))(\varphi) = \varphi(f(x))$$

Therefore, $f^{**}(\eta_X(x))(\varphi) = \eta_Y(f(x))(\varphi)$ for all $x \in X, \varphi \in Y^*$. Hence $f^{**}(\eta_X(x)) = \eta_Y(f(x))$ for all $x$, and $f^{**}\eta_X = \eta_Y f$. Therefore, $\eta$ is a natural transformation.

The foregoing can be done with modules as well as monoid actions.

If $F, G : \mathbf{C} \to \mathbf{D}$ are contravariant functors, one can still define a natural transformation from $F$ to $G$. This time, it's an assignment $A \to \eta_A$ with $\eta_A \in \hom_{\mathbf{D}}(FA, GA)$, such that for all $f : A \to B$ in $\mathbf{C}$,

$$\begin{array}{ccc} FB & \xrightarrow{\eta_B} & GB \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FA & \xrightarrow{\eta_A} & GA \end{array}$$

is commutative. This need not be studied seperately, though, because $F$ and $G$ are actually covariant functors from $\mathbf{C}^{\text{op}}$ to $\mathbf{D}$.

Let $F, G, H : \mathbf{C} \to \mathbf{D}$ be functors, $\eta : F \Rightarrow G$ and $\zeta : G \Rightarrow H$ natural transformations. Then it is clear that the assignment $A \to \zeta_A\eta_A$ is a natural transformation from $F$ to $H$. It is notated as $\zeta\eta$ and is called the **composite natural transformation**. Also, we have the **identity natural transformation** $1_F$ from $F$ to $F$ assigning $A \to 1_{FA}$. Evidently $(\theta\zeta)\eta = \theta(\zeta\eta)$ and $1_G\eta = \eta = \eta 1_F$ when they are defined. It therefore follows that the covariant functors from $\mathbf{C}$ to $\mathbf{D}$ form a category, whose morphisms are natural transformations of functors. [We temporarily allow $\hom(F, G)$ to be a proper class here.] It is called the **functor category** of $\mathbf{C}$ to $\mathbf{D}$ and is notated $\mathbf{D}^{\mathbf{C}}$.

Evidently in $\mathbf{D}^{\mathbf{C}}$, isomorphisms are natural isomorphisms of functors.

**EXERCISES**

1. Let $F : \mathbf{C} \to \mathbf{D}$ be a covariant functor.

   (a) $F$ preserves isomorphisms; that is, if $f : A \to B$ is an isomorphism in $\mathbf{C}$, then $F(f) : FA \to FB$ is an isomorphism in $\mathbf{D}$.

   (b) $F$ also preserves sections and retractions.

   (c) If $F(f)$ is monic and $F$ is faithful, then $f$ is monic. Give a counterexample when $F$ is not faithful.

   (d) If $f$ is monic and $F$ is full, then $F(f)$ is monic. Give a counterexample when $F$ is not full.

   (e) Repeat parts (c) and (d) with "monic" replaced with "epic".

   (f) What if $F$ is contravariant? Modify parts (a)-(e) to hold for contravariant $F$.

2. If $\mathbf{Q}$ is any class of objects which assigns any two objects $A, B$ a set $\hom(A, B)$, but there's no notion of composition of morphisms or identity morphisms, $\mathbf{Q}$ is called a **quiver**. Thus a category is a quiver when that information is disregarded in it.

   If $\mathbf{Q}$ and $\mathbf{R}$ are quivers, a map from $\mathbf{Q}$ to $\mathbf{R}$ is kind of like a functor without the structure to preserve: it assigns each $A \in \text{ob}(\mathbf{Q})$ an object $FA$ in $\mathbf{R}$ and each $f : A \to B$ in $\mathbf{Q}$ a map $F(f) : FA \to FB$ in $\mathbf{R}$.

   (a) For each quiver $\mathbf{Q}$, the define the **free category** $\mathbf{C}$ given by $\mathbf{Q}$ as follows: $\text{ob}(\mathbf{C}) = \text{ob}(\mathbf{Q})$, and for any objects $A, B$, $\hom_{\mathbf{C}}(A, B)$ is the set of all strings of the form $f_n \dots f_2 f_1$ with $f_1 : A \to A_1, f_2 : A_1 \to A_2, \dots f_n : A_{n-1} \to B$ in $\mathbf{Q}$. When $A = B$, the empty string is included as $1_A$. Define the composition of morphisms by the obvious juxtaposition. Verify that $\mathbf{C}$ is indeed a category.

   (b) Let $I : \mathbf{Q} \to \mathbf{C}$ be the map sending objects to themselves and morphisms to themselves as one-element strings. If $\mathbf{D}$ is a category and $J : \mathbf{Q} \to \mathbf{D}$ is a map, there is a unique functor $F : \mathbf{C} \to \mathbf{D}$ such that $J = FI$.

3. Let $U$ be the functor $\mathbf{Ring} \to \mathbf{Grp}$ of Example 6, and $M_n$ the functor from $\mathbf{Ring} \to \mathbf{Ring}$ of Example 8. Let $GL_n$ be the composite functor $UM_n$; to what does it send a ring $R$?

4. Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties, and $F, G$ the functors given in Examples 1 and 12.

   (a) For each $A \in \mathcal{V}(S_1)$, regard $A$ as a $\mathcal{V}(S_2)$-algebra with the derived structure, and let $(U, i)$ be a universal enveloping $A$ for the takeoff $T$. Show that $i : A \to U$ has a unique retraction $r_A : U \to A$ which is an $\Omega_1$-homomorphism.

   (b) Explain why $r_A \in \hom_{\mathbf{V}(S_1)}(GFA, A)$.

   (c) Show that $A \to r_A$ is a natural transformation from $GF$ to $1_{\mathbf{V}(S_1)}$.

5. (a) Give two examples of covariant functors $\mathbf{V}(S) \to \mathbf{V}(S)-\mathbf{sub}$. [$\mathbf{V}(S)-\mathbf{sub}$ is defined in Exercise 7 of Section 1.]

   (b) Give two examples of covariant functors $\mathbf{V}(S)-\mathbf{sub} \to \mathbf{V}(S)$.

   (c) Do parts (a) and (b) for $\mathbf{V}(S)-\mathbf{con}$.

   (d) Now express canonical monomorphisms and canonical epimorphisms in the form of natural transformations.

6. Let $\mathbf{C}$ be a category, $A$ an object of $\mathbf{C}$. Let $\mathbf{C}/A$ be the category in Example 11 of Section 1. Define $F : \mathbf{C}/A \to \mathbf{C}$ sending $(B, f) \to B$ and each morphism $u$ to itself. Then $F$ is a functor.

7. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors, and suppose for each $A \in \text{ob}(\mathbf{C})$, $\eta_A : FA \to GA$ is any morphism. [This is called an **infranatural transformation**.] Show that $\mathbf{C}_\eta$ is a subcategory of $\mathbf{C}$ when defined by $\text{ob}(\mathbf{C}_\eta) = \text{ob}(\mathbf{C})$, and each $f \in \hom_{\mathbf{C}}(A, B)$ is in $\hom_{\mathbf{C}_\eta}(A, B)$ if and only if

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow{\eta_B} & GB \end{array}$$

   is commutative. [$\mathbf{C}_\eta$ is referred to as the **naturalizer** of $\eta$.]

8. Let $F, G : \mathbf{C} \to \mathbf{D}, H : \mathbf{D} \to \mathbf{E}, K : \mathbf{B} \to \mathbf{C}$ be functors. Let $\eta : F \Rightarrow G$ be a natural transformation.

   (a) $H\eta : HF \Rightarrow HG$ is a natural transformation given by $A \to H(\eta_A)$ for $A \in \text{ob}(\mathbf{C})$.

   (b) $\eta K : FK \Rightarrow GK$ is a natural transformation given by $A \to \eta_{KA}$ for $A \in \text{ob}(\mathbf{B})$.

   Now assume capital letters are functors and lowercase Greek letters are natural transformations.

   (c) The products are *functorial*:

$$F(\zeta\eta) = (F\zeta)(F\eta), F1_G = 1_{FG}, (\zeta\eta)G = (\zeta G)(\eta G), 1_F G = 1_{FG}$$

   when they are defined. [*Hint*: Just use the definition!]

   (d) The products form a *biaction*:

$$(GF)\eta = G(F\eta), 1_{\mathbf{D}}\eta = \eta, \eta(GF) = (\eta G)F, \eta 1_{\mathbf{C}} = \eta, (G\eta)F = G(\eta F)$$

   when they are defined.

   (e) Suppose $F, F' : \mathbf{C} \to \mathbf{D}$, $G, G' : \mathbf{D} \to \mathbf{E}$ are functors and $\eta : F \Rightarrow F'$, $\zeta : G \Rightarrow G'$ are natural transformations. Show that $(\zeta F')(G\eta) = (G'\eta)(\zeta F)$ as natural transformations $GF \Rightarrow G'F'$. [*Hint*: This comes from the naturality of one of them.]

9. Express Example 9 of Section 1 in the form of a functor category.

10. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $A \in \text{ob}(\mathbf{C})$ is fixed, then $P : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$ given by $F \to FA, \eta \to \eta_A$ is a functor. It is called the **projection onto $A$ of the functor category**.

11. Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be categories, $F : \mathbf{D} \to \mathbf{E}$ a functor. Use Exercise 8 to define functors $F^{\mathbf{C}} : \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}$ and $\mathbf{C}^F : \mathbf{C}^{\mathbf{E}} \to \mathbf{C}^{\mathbf{D}}$.

12. (a) A functor $F : \mathbf{C} \to \mathbf{D}$ is an **isomorphism** provided there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $GF = 1_{\mathbf{C}}$ and $FG = 1_{\mathbf{D}}$. Informally, what can you say about isomorphic categories?

    (b) Let $\cong$ denote natural isomorphism of functors here. If $F \cong G$, then $HF \cong HG$ and $FK \cong GK$ when they are defined. [*Hint*: Exercise 11 gives a shortcut.]

    (c) A functor $F : \mathbf{C} \to \mathbf{D}$ is an **equivalence** if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$. $\mathbf{C}$ and $\mathbf{D}$ are **equivalent** if such a functor $F$ exists. Show that this is an equivalence relation on the categories. Also, isomorphic categories are equivalent, but not conversely.

    (d) A functor $F : \mathbf{C} \to \mathbf{D}$ is an equivalence if and only if $F$ is faithful and full and for every $B \in \text{ob}(\mathbf{D})$ there exists $A \in \text{ob}(\mathbf{C})$ such that $FA$ and $B$ are isomorphic in $\mathbf{D}$. [*Hint*: $\Rightarrow$ If $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$, then $FG$ and $GF$ are faithful and full [why?] Use this to prove that $F$ and $G$ are faithful and full. Also show that for $B \in \text{ob}(\mathbf{D})$, $FGB$ is isomorphic to $B$. $\Leftarrow$ Define $G : \mathbf{D} \to \mathbf{C}$ sending each $B$ to some $A$ such that there exists an isomorphism $\sigma_B : FA \to B$. Show that there is a unique way for $G$ to assign morphisms so that $\sigma$ is a natural isomorphism $FG \cong 1_{\mathbf{D}}$. To show $GF \cong 1_{\mathbf{C}}$, note that for each $A \in \text{ob}(\mathbf{C})$ there is a unique isomorphism $\eta_A : GFA \to A$ such that $F(\eta_A) = \sigma_{FA}$.]

13. Recall that if $A, B, C$ are $\mathcal{V}(S)$ algebras, $(A \times B) \times C \cong A \times (B \times C)$ due to an isomorphism $\sigma_{A,B,C}$ sending $((a,b),c) \to (a,(b,c))$. Show that this isomorphism is *natural* in the sense that for all $f : A \to A', g : B \to B', h : C \to C'$ this diagram is commutative:

$$\begin{array}{ccc} (A \times B) \times C & \xrightarrow{\sigma_{A,B,C}} & A \times (B \times C) \\ {\scriptstyle (f\times g)\times h}\downarrow & & \downarrow{\scriptstyle f\times(g\times h)} \\ (A' \times B') \times C' & \xrightarrow{\sigma_{A',B',C'}} & A' \times (B' \times C') \end{array}$$

    Here $f \times g$ denotes the product of maps; that is, $(f \times g)(a,b) = (f(a), g(b))$.

14. Define the **center** of a category $\mathbf{C}$ to be the class of natural transformations from $1_{\mathbf{C}}$ to $1_{\mathbf{C}}$; that is, $\hom(1_{\mathbf{C}}, 1_{\mathbf{C}})$ in the functor category $\mathbf{C}^{\mathbf{C}}$. Evidently the center has a monoidal structure under composition of natural transformations.

Now let $M$ be a fixed monoid, $C(M) = \{a \in M \mid ax = xa \ \forall x \in M\}$ be the center of $M$. For $c \in C(M)$, the assignment of each $M$-action $X$ to the homomorphism $x \to cx$ from $X \to X$ is a natural transformation $\eta(c)$ from $1_{M-\mathbf{act}}$ to itself. Furthermore, the map $c \to \eta(c)$ is a monoid isomorphism from $C(M)$ into the center of $M-\mathbf{act}$.

# 2.4 - Products and Coproducts

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

Objects in a category can be combined in interesting ways. Two of the convenient operators combining them are products and coproducts. To see what they are like, consider $\mathcal{V}(S)$ algebras.

If the $A_\alpha$ are $\mathcal{V}(S)$ algebras, one can take $A = \Pi A_\alpha$, along with the projection homomorphisms $p_\alpha : A \to A_\alpha$ from the product.

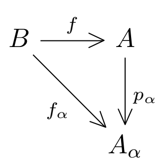
Now suppose $B$ is a $\Omega$-algebra and $f_\alpha : B \to A_\alpha$ is a homomorphism for each $\alpha$. Define $f : B \to A$ so that $f(b)_\alpha = f_\alpha(b)$. That determines $f(b)$ for each $b$, and it is seen that $f$ is the only homomorphism $B \to A$ such that $f_\alpha = p_\alpha f$ for all $\alpha$. This illustrates a product in terms of purely homomorphisms:

**Whenever $f_\alpha : B \to A_\alpha$ is a homomorphism for each $\alpha$, there is a unique homomorphism $f : B \to \Pi A_\alpha$ such that $f_\alpha = p_\alpha f$ for all $\alpha$.**

This property leads to the following definition in category theory.

**DEFINITION**

*Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$ [with possible repetitions]. A **product** of the $A_\alpha$'s is a pair $(A, \{p_\alpha\})$ with $A \in ob(\mathbf{C})$ and $p_\alpha : A \to A_\alpha$ for each $\alpha$, such that whenever $B \in ob(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$ for each $\alpha$, there is a unique morphism $f : B \to A$ such that for all $\alpha$ we have $p_\alpha f = f_\alpha$; in other words,*




*is commutative.*

The fact that the definition says "*a* product," rather than "*the* product," can be remedied, as Exercise 1 shows that products are unique up to a unique isomorphism.

**EXAMPLES**

1. We have just shown that products in $\mathbf{V}(S)$ coincide with the product of algebras in Chapter 1, Section 2.

2. If $\{(A_\alpha, B_\alpha)\}$ are objects in $\mathbf{V}(S)-\mathbf{sub}$, notice that with each $B_\alpha$ a subalgebra of $A_\alpha$, $\Pi B_\alpha$ is a subalgebra of $\Pi A_\alpha$. We claim that $(\Pi A_\alpha, \Pi B_\alpha)$, along with the usual projections $p_\alpha : \Pi A_\alpha \to A_\alpha$, is a product of the objects in $\mathbf{V}(S)-\mathbf{sub}$. To begin with, the $p_\alpha$'s are admitted by the category since $p_\alpha$ sends elements of $\Pi B_\alpha$ to elements of $B_\alpha$. Now suppose $f_\alpha : (C, C_1) \to (A_\alpha, B_\alpha)$ are morphisms. This requires that each $f_\alpha$ is a homomorphism $C \to A_\alpha$ satisfying

$f_\alpha(C_1) \subseteq B_\alpha$. With that, if $f : C \to A_\alpha$ is the coordinate map [$p_\alpha f = f_\alpha$ for each $\alpha$], $f(C_1) \subseteq \Pi B_\alpha$. This means $f$ is a morphism $(C, C_1) \to (\Pi A_\alpha, \Pi B_\alpha)$, and is clearly the only one satisfying $p_\alpha f = f_\alpha$. This proves our claim.

3. Let $\{(A_\alpha, \Phi_\alpha)\}$ be objects in $\mathbf{V}(S)-\mathbf{con}$. For $a, b \in \Pi A_\alpha$, define $a \Phi b$ if $a_\alpha \Phi_\alpha b_\alpha$ for all $\alpha$. Then this is a congruence relation on $\Pi A_\alpha$, and an argument similar to the one above shows that $(\Pi A_\alpha, \Phi)$ is a product of the objects in $\mathbf{V}(S)-\mathbf{con}$.

4. Products in **Cat** are product categories with the projection functors.

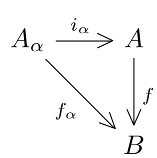
5. If $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ are varieties, one can form a new variety $\mathcal{V}(S_3)$ taking disjoint unions of operators and identities. That is, $\Omega_3(n) = \Omega_1(n) \uplus \Omega_2(n)$ for $n \geq 0$ and $S_3 = S_1 \uplus S_2$. Then a $\mathcal{V}(S_3)$ algebra is precisely a set with both a $\mathcal{V}(S_1)$ structure and a $\mathcal{V}(S_2)$ structure which are independent of one another. Takeoffs $\mathcal{V}(S_3) \to \mathcal{V}(S_1), \mathcal{V}(S_2)$ can be formed, each dropping one of the structures, and $\mathcal{V}(S_3)$ is a product of $\mathcal{V}(S_1)$ and $\mathcal{V}(S_2)$ in the category **Var**.

6. What does it mean for an object $T$ to be a product of the empty batch $\{\}$? Well, there are no $p_\alpha$'s involved in this case, and whenever $B \in \mathrm{ob}(\mathbf{C})$ [there are no $f_\alpha$'s involved], there is a unique morphism $f : B \to T$ [no diagram commutativity is needed]. Stated otherwise, for all $B \in \mathrm{ob}(\mathbf{C})$, $\hom(B, T)$ consists of a single element; in other words, $T$ is a terminal object.

Coproducts are basically the dual of products, and in fact, we have already started them in Section 9 of Chapter 1. They carry over to category theory.

**DEFINITION**

*Let $\{A_\alpha\}$ be a batch of objects in a category $\mathbf{C}$ [with possible repetitions]. A **coproduct** of the $A_\alpha$'s is a pair $(A, \{i_\alpha\})$ with $A \in ob(\mathbf{C})$ and $i_\alpha : A_\alpha \to A$ for each $\alpha$, such that whenever $B \in ob(\mathbf{C})$ and $f_\alpha : A_\alpha \to B$ for each $\alpha$, there is a unique morphism $f : A \to B$ such that for all $\alpha$ we have $f i_\alpha = f_\alpha$; in other words,*




*is commutative.*

Coproducts are basically products in the opposite category $\mathbf{C}^{\mathrm{op}}$. Coproducts in $\mathbf{V}(S)$ coincide with the definition of a coproduct in Section 9 of Chapter 1. There, we proved that coproducts always exist in $\mathbf{V}(S)$, and here we shall use the proof to derive a subtle and interesting explanation on how to find them.

1. Suppose you are given a batch $\{A_\alpha\}$. Let $F$ be the free algebra given by the set $\uplus A_\alpha$ with *set map* $i : \uplus A_\alpha \to F$. Then form $j_\alpha : A_\alpha \to F$ for each $\alpha$ by composing $i$ with each injection $A_\alpha \to \uplus A_\alpha$.

2. To make each $j_\alpha$ a homomorphism, identify any expression in $F$ whose symbols come from a single $A_\alpha$ with its value given by $A_\alpha$. That is, identify

$(\omega j_\alpha(a_1) j_\alpha(a_2) \ldots j_\alpha(a_n))$ with $j_\alpha(\omega a_1 a_2 \ldots a_n)$. To do this, find the congruence relation $\Theta$ on $F$ generated by those pairs and let $\pi : F \to F/\Theta$ be the canonical epimorphism. *Make no more identifications than that* or it won't work.

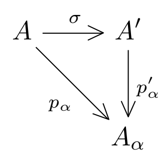
3. Then each $\pi j_\alpha$ is a homomorphism and $(F/\Theta, \pi j_\alpha)$ is a coproduct of the $A_\alpha$'s.

To make a long story short, the coproduct of algebras consists of expressions whose symbols are in all the algebras, such that the identities in $S$ are satisfied, and any expression with its symbols in a single algebra is identified with the value the algebra gives it. You can give each algebra a different color to see this easily. Operator symbols and parentheses have no color.

In the case of groups, this precise procedure gives the familiar free product on groups; same for monoids. Also, it gives $R$-modules their direct sum, and commutative rings their *tensor product* [to be learned later].

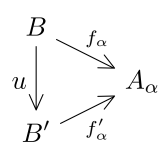
**EXERCISES**

1. Let $\{A_\alpha\}$ be a batch of objects in a category **C**. If $(A, \{p_\alpha\})$ and $(A', \{p'_\alpha\})$ are both products of the $A_\alpha$'s in **C**, there is a unique isomorphism $\sigma : A \to A'$ such that for all indices $\alpha$ the diagram




   is commutative. Dualize.

2. Let $\{A_\alpha\}$ be a batch of objects in a category **C**. Define $\mathbf{C}/\{A_\alpha\}$ as follows: $\mathrm{ob}(\mathbf{C}/\{A_\alpha\})$ is the class of pairs of the form $(B, \{f_\alpha\})$ where $B \in \mathrm{ob}(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$ for each $\alpha$, and $\hom((B, \{f_\alpha\}), (B', \{f'_\alpha\}))$ is the set of morphisms $u : B \to B'$ such that for every $\alpha$




   is commutative. Define composition of morphisms and identity morphisms as in **C**. Verify that this data form a category, and that a product of the $A_\alpha$'s is a terminal object in $\mathbf{C}/\{A_\alpha\}$. Dualize.

3. If $T$ is a terminal object of category **C**, show that $A$ is a product of $T$ and $A$ and determine the projection maps.

4. If $I$ is an initial object of a category **C**, show that $A$ is a coproduct of $I$ and $A$.

5. If **C** is a category given by a preorder [Example 7 of Section 1], describe products of objects in **C**.

6. (a) Suppose **C** is a category in which any two objects in **C** have a product in **C**. [Such a category is called a **category with finite products**.] Show that any finite batch of objects in **C** has a product in **C**.

   (b) Give an example of a category **C** with finite products having an infinite batch of objects with no product.

7. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be morphisms in a category **C**. Suppose $(A, p_1, p_2)$ is a product of $A_1$ and $A_2$ and $(B, q_1, q_2)$ is a product of $B_1$ and $B_2$.

   (a) There is a unique morphism $f : A \to B$ such that the two rectangles in

$$\begin{array}{ccccc} A_1 & \xleftarrow{p_1} & A & \xrightarrow{p_2} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xleftarrow{q_1} & B & \xrightarrow{q_2} & B_2 \end{array}$$

   are commutative.

   (c) Suppose **C** is a category with finite products. Define $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ as follows: Assign each object $(A_1, A_2)$ to a product of $A_1$ and $A_2$, and each morphism $(f_1, f_2)$ to the morphism $f$ given in part (a). Verify that $F$ is a functor. It is called the *product-giving functor* for **C**.

   (d) Any two functors defined in the way of (c) are naturally isomorphic.

# 2.5 - Universals

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

The universals learned in Section 1.11 can be generalized to any functor of categories. Firstly, if $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ is a takeoff of varieties, recall what a universal enveloping $B \in \mathcal{V}(S_2)$ is: it consists of a pair $(U, u)$ with $U \in \mathcal{V}(S_1)$ and $u : B \to U$ an $\Omega_2$-homomorphism, such that whenever $(A, f)$ is another such pair, there is a unique $\Omega_1$-homomorphism $h : U \to A$ such that

$$\begin{array}{ccc} B & \xrightarrow{u} & U \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle h} \\ & & A \end{array}$$

is commutative.

This leads to the following definition. Exercise care in the fact that the statement $f = hu$ treats $U, A$ and $h$ as they are in $\mathbf{V}(S_2)$ when they are virtuously in $\mathbf{V}(S_1)$.

**DEFINITION**

*Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, $B \in ob(\mathbf{D})$. A **universal from** $B$ **to** $F$ is a pair $(U, u)$ with $U \in ob(\mathbf{C})$ and $u \in \hom_{\mathbf{D}}(B, FU)$ such that whenever $(A, f)$ is another pair with $A \in ob(\mathbf{C})$ and $f \in \hom_{\mathbf{D}}(B, FA)$ there exists a unique $h \in \hom_{\mathbf{C}}(U, A)$ such that the diagram*

$$\begin{array}{ccc} B & \xrightarrow{u} & FU \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle F(h)} \\ & & FA \end{array}$$

*is commutative. $U$ is called the **universal object** and $u$ is called the **universal map**.*

**EXAMPLES**

1. A takeoff of varieties becomes a functor, and the definition of a universal for that functor coincides with the universal learned in Section 1.11.

In the special case where $T$ is the unique takeoff from $\mathcal{V}(S)$ to the variety of sets, $F : \mathbf{V}(S) \to \mathbf{Set}$ is the forgetful functor, and a universal from a set $X$ to $F$ is $F_S(\Omega, X)$, the free $\mathcal{V}(S)$-algebra given by $X$.

2. Let **Dom** and **Field** be the categories of integral domains and fields, respectively. Then let $\mathbf{Dom}^m$ be the subcategory of **Dom** keeping all the objects but only the *monomorphisms.* Since every field is an integral domain and every homomorphism of fields is injective, one can form a functor $F : \mathbf{Field} \to \mathbf{Dom}^m$ sending every field and morphism to itself. For any integral domain $R$, let $K$

be the field of quotients of $R$, with injection $i : R \to K$. Then $(K, i)$ is easily seen to be a universal from $R$ to $F$.

3. Let $\mathbf{C}$ be any category and $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ be the diagonal functor given in Example 14 of Section 3. Then a universal from $(B_1, B_2) \in \text{ob}(\mathbf{C} \times \mathbf{C})$ to $\Delta$ is a coproduct of $B_1$ and $B_2$. To see this, the universal takes the form $(U, u)$ with $u : (B_1, B_2) \to \Delta U$, that is, [since $\Delta U = (U, U)$], $u$ is a pair of morphisms $u_1 : B_1 \to U, u_2 : B_2 \to U$. The additional property is satisfied that whenever $f : (B_1, B_2) \to \Delta A$, that is, $f$ is a pair of morphisms, $f_1 : B_1 \to A, f_2 : B_2 \to A$, there is a unique morphism $h : B \to A$ such that $f = \Delta(h)u$. Since $\Delta(h) = (h, h)$, this says the same thing as $f_1 = hu_1, f_2 = hu_2$. Therefore, $(U, u_1, u_2)$ is a coproduct of $B_1$ and $B_2$.

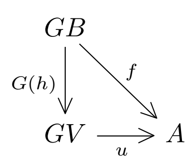
This generalizes to coproducts of more than two objects.

Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor and $B \in \text{ob}(\mathbf{D})$. The proof of Theorem 1.28 applies here, showing that if $(U, u)$ and $(U', u')$ are both universals from $B$ to $F$, there exists a unique isomorphism $\sigma : U \to U'$ such that $i' = F(\sigma)i$. Thus universals are unique up to a unique isomorphism. There is also a "composition law" for universals; see Exercise 1.

As expected, there is a dual to the definition obtained by reversing the arrows:

**DEFINITION**

*Let $G : \mathbf{D} \to \mathbf{C}$ be a functor, $A \in ob(\mathbf{C})$. A* ***universal from*** *$G$* ***to*** *$A$ is a pair $(V, v)$ with $V \in ob(\mathbf{D})$ and $v \in \hom_{\mathbf{C}}(GV, A)$ such that whenever $(B, f)$ is another pair with $B \in ob(\mathbf{D})$ and $f \in \hom_{\mathbf{C}}(GB, A)$ there exists a unique $h \in \hom_{\mathbf{D}}(B, V)$ such that the diagram*




*is commutative. $V$ is called the* ***universal object*** *and $v$ is called the* ***universal map****.*

**EXAMPLES**

1. Let $\mathcal{V}(S)$ be a variety and $G : \mathbf{Set} \to \mathbf{V}(S)$ be the free-algebra functor [Example 12 of Section 3]. If $A \in \mathcal{V}(S)$, let $V$ be the *set* $A$. By virtue of a free algebra, the identity map $V \to A$ [they are the same set, but the codomain is regarded as an algebra] extends to the **evaluation homomorphism** $v : F_S(\Omega, V) \to A$. We claim that $(V, v)$ is a universal from $G$ to $A$. To see this, let $(B, f)$ be another pair with $B$ a set and $f : GB \to A$ a homomorphism. Then, composing with the inclusion $B \to GB$ yields a unique set map $h : B \to V$ [$V$ is the set $A$]. Checking on symbols shows that $uG(h) = f$ and $h$ is unique for this property.

This generalizes to the functor $\mathbf{V}(S_2) \to \mathbf{V}(S_1)$ induced by a takeoff $\mathcal{V}(S_1) \to \mathcal{V}(S_2)$ in Example 12 of Section 3. We leave it to the reader to carry out the details.

2. Let $M$ be a fixed monoid and $G : M{-}\mathbf{act} \to \mathbf{Set}$ be the forgetful functor. If $X$ is a set, we define the **power action** as follows: $X^M$ is the set of all functions from the monoid $M$ to $X$, and for $\varphi \in X^M$ and $n \in M$, $n\varphi$ is the map $m \to \varphi(mn)$ from $M \to X$. It is straightforward that this makes $X^M$ a power action. Furthermore, the *projection* $p : X^M \to X$ sending $\varphi \to \varphi(1)$ can be considered. We claim that $(X^M, p)$ is a universal from $G$ to $X$. Thus this functor possesses both kinds of universals.

Suppose $Y$ is any $M$-action and $f : Y \to X$ is a set map. Then define $h : Y \to X^M$ by assigning $h(y)$ to the map $m \to f(my)$ from $M \to X$. Thus $h(y)(m) = f(my)$. We need to show three things:

(i) $h$ is a homomorphism;

(ii) $ph = f$ as maps $Y \to X$;

(iii) $h$ is unique for properties (i) and (ii).

To show (i), note that for $n \in M$, $h(ny)$ is the map $m \to f(mny)$. On the other hand, $nh(y)$ — by definition of the power action — is the map from $m \to h(y)(mn) = f((mn)y) = f(mny)$. Therefore, $h(ny)$ an $nh(y)$ are equal, so that $h$ is a homomorphism.

(ii) is easy to show because for all $y \in Y$, $ph(y) = p(h(y)) = h(y)(1) = f(1y) = f(y)$. To show (iii), suppose $h' : Y \to X^M$ is also a homomorphism satisfying $ph' = f$. Then for all $y \in Y, m \in M$,

$$h'(y)(m) = h'(y)(1m) = mh'(y)(1) = h'(my)(1) = p(h'(my)) = ph'(my) = f(my)$$

Therefore $h'(y)$ is necessarily the map $m \to f(my)$ for all $y \in Y$, so that $h' = h$ and $h$ is uniquely determined.

**EXERCISES**

1. Let $F_1 : \mathbf{C} \to \mathbf{D}, F_2 : \mathbf{D} \to \mathbf{E}$ be functors, and $B \in \text{ob}(\mathbf{E})$. If $(U_2, u_2)$ is a universal from $B$ to $F_2$ and $(U_1, u_1)$ is a universal from $U_2$ to $F_1$, prove that $(U_1, F_2(u_1)u_2)$ is a universal from $B$ to $F_2F_1$.

2. Let $\mathbf{C}$ be any category and $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ be the diagonal functor. Then a universal from $\Delta$ to $(B_1, B_2) \in \text{ob}(\mathbf{C} \times \mathbf{C})$ is a product of $B_1$ and $B_2$.

3. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, $B \in \text{ob}(\mathbf{D})$. Define a category $\mathbf{D}(B, F)$ as follows: the objects of $\mathbf{D}(B, F)$ are the pairs of the form $(A, f)$ with $A \in \text{ob}(\mathbf{C})$ and $f : B \to FA$. If $(A_1, f_1), (A_2, f_2)$ are objects, a morphism $(A_1, f_1) \to (A_2, f_2)$ in $\mathbf{D}(B, F)$ is an arrow $g : A_1 \to A_2$ such that $f_2 = F(g)f_1$. Verify that this data forms a category, and that a universal from $B$ to $F$ is an initial object of $\mathbf{D}(B, F)$. Dualize.

# 2.6 - Limits and Colimits

Nicholas McConnell

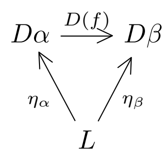
(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

**This section is not a prerequisite of any other and may be skipped if desired.**

Products, pullbacks and difference kernels are all examples of a more general concept called the **limit**. To see this, let $\mathbf{J}$ be an "elementary" category [such as a discrete category], and $D : \mathbf{J} \to \mathbf{C}$ a functor. Then $D$ is called a **diagram of type J** in $\mathbf{C}$.

Define a **cone** to the diagram $D$ to be a natural transformation from a constant functor onto an object $L$ in $\mathbf{C}$ [Example 5 of Section 3] to $D$. Stated otherwise, it is a pair $(L, \{\eta_\alpha\})$ with $L \in \text{ob}(\mathbf{C})$, $\eta_\alpha : L \to D\alpha$ with $\alpha \in \text{ob}(\mathbf{J})$, such that for every $f \in \hom_{\mathbf{J}}(\alpha, \beta)$ the following diagram is commutative:




Given this data we define

**DEFINITION**

*Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram of type $\mathbf{J}$ in $\mathbf{C}$. A **limit** of $D$ is defined to be a cone $(L, \{\eta_\alpha\})$ to $D$ such that for every cone $(B, \{\zeta_\alpha\})$ to $D$, there exists a unique morphism $\theta : B \to L$ such that $\zeta_\alpha = \eta_\alpha \theta$ for all $\alpha$.*

Again it is routine to show the "uniqueness up to a unique isomorphism": if $(L, \{\eta_\alpha\})$ and $(L', \{\eta'_\alpha\})$ are both limits of $D$, there is a unique isomorphism $L' \to L$ such that $\eta'_\alpha = \eta_\alpha \sigma$ for all $\alpha$. For one, this is an immediate consequence of limits being universals for suitably defined categories. See Exercise 1.

**EXAMPLES**

1. If $\mathbf{J}$ is a discrete category, then its only morphisms are the identity morphisms, and the commutativity of the diagram in the definition of a cone is a tautology. Thus a cone is simply a pair $(L, \{\eta_\alpha\})$ with $L \in \text{ob}(\mathbf{C})$ and $\eta_\alpha : L \to D\alpha$. It is easy to see that a limit of $D$ is simply a product of the $D\alpha$'s [Section 4].

2. Let $\mathbf{J}$ be the category with only two objects $\alpha, \beta$ such that the only morphisms are the identity morphisms and two morphisms $\alpha \to \beta$. Then a diagram of type $\mathbf{J}$ in $\mathbf{C}$ is a pair of objects $A_\alpha, A_\beta \in \text{ob}(\mathbf{C})$ with two morphisms $f_1, f_2 : A_\alpha \to A_\beta$. A cone is a triple $(L, \eta_\alpha, \eta_\beta)$ such that $\eta_\beta = f_1 \eta_\alpha$ and $\eta_\beta = f_2 \eta_\alpha$, or is identifiably a pair $(L, \eta)$ with $f_1 \eta = f_2 \eta$. Thus a limit is a difference kernel [or equalizer] [see Exercise 2 of Section 2]. This can be done for more than two morphisms as well.

3. Suppose $\mathbf{J}$ has three objects, $\alpha, \beta, \gamma$ and two nonidentity morphisms, $\alpha \to \gamma$ and $\beta \to \gamma$. Then a diagram of type $\mathbf{J}$ in $\mathbf{C}$ is a triple of objects $A_1, A_2, A \in \mathrm{ob}(\mathbf{C})$ with two morphisms $f_1 : A_1 \to A, f_2 : A_2 \to A$. A cone is identifiably a pair $(L, \eta_1, \eta_2)$ such that $f_1\eta_1 = f_2\eta_2$ and a limit is therefore a pullback in this case.

One of the things which makes the category $\mathbf{V}(S)$ so special is that it contains all limits. That is, every diagram in $\mathbf{V}(S)$ of any type has a limit, as we now prove. [Such a category is called a **complete category**.]

**THEOREM 2.3** *Limits exist in* $\mathbf{V}(S)$ *for any diagram* $D : \mathbf{J} \to \mathbf{V}(S)$.

*Proof of Theorem 2.3.* Let $D : \mathbf{J} \to \mathbf{V}(S)$ be a diagram. Then define $A$ to be the following subset of the product $\prod_{\alpha \in \mathrm{ob}(\mathbf{J})} D\alpha$:

$$A = \{a \in \prod_{\alpha \in \mathrm{ob}(\mathbf{J})} D\alpha \mid D(f)(a_\alpha) = a_\beta \ \forall f \in \hom_{\mathbf{J}}(\alpha, \beta)\}$$

We claim that $A$ is a subalgebra of the product. Suppose $\omega \in \Omega(0)$, then $(\omega_{\Pi D\alpha}) \in A$ because $D(f)(\omega_{D\alpha}) = (\omega_{D\beta})$ for all $f : \alpha \to \beta$. Now suppose $n \geq 1, \omega \in \Omega(n)$ and $a^1, a^2, \ldots a^n \in A$. Then for all $f : \alpha \to \beta$,

$$D(f)((\omega a^1 a^2 \ldots a^n)_\alpha) = D(f)(\omega a^1_\alpha a^2_\alpha \ldots a^n_\alpha) = (\omega D(f)(a^1_\alpha) D(f)(a^2_\alpha) \ldots D(f)(a^n_\alpha))$$
$$= (\omega a^1_\beta a^2_\beta \ldots a^n_\beta) = (\omega a^1 a^2 \ldots a^n)_\beta$$

Therefore, $(\omega a^1 a^2 \ldots a^n) \in A$, and $A$ is a subalgebra. Now let $\eta_\alpha : A \to D\alpha$ be the *restricted projections* [that is, $p_\alpha(a) = a_\alpha$ for $a \in A$]. Then $(A, \{\eta_\alpha\})$ is a cone to $D$ because the $\eta_\alpha$ are homomorphisms and for all $f : \alpha \to \beta$ in $\mathbf{J}$,

$$\eta_\beta(a) = a_\beta = D(f)(a_\alpha) = D(f)\eta_\alpha(a)$$

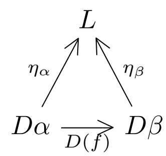
Therefore $\eta_\beta = D(f)\eta_\alpha$.

Now suppose $(B, \{\zeta_\alpha\})$ is any cone to $D$. Then $\zeta_\alpha : B \to D\alpha$ and one can form the coordinate map $\zeta : B \to \Pi D_\alpha$ satisfying $\zeta(b)_\alpha = \zeta_\alpha(b)$. We claim that im $\zeta \subseteq A$, so that $\zeta$ can be surjectified into a homomorphism $\theta : B \to A$. To see this, use the fact that $(B, \{\zeta_\alpha\})$ is a *cone*, and hence

$$D(f)(\zeta(a)_\alpha) = D(f)(\zeta_\alpha(a)) = D(f)\zeta_\alpha(a) = \zeta_\beta(a) = \zeta(a)_\beta$$

Therefore, $\theta$ exists and, and obviously $\zeta_\alpha = \eta_\alpha\theta$. Since an element of $A$ is completely determined by where each $\eta_\alpha$ sends it, $\theta$ is unique. Therefore, $(A, \{\eta_\alpha\})$ is a limit. ■

Colimits are the dual of limits, and they are obtained by reversing the arrows. This doesn't mean make the functor $\mathbf{J} \to \mathbf{C}$ contravariant, though [which could be remedied anyway, by changing $\mathbf{J}$ into $\mathbf{J}^{\mathrm{op}}$]. If $D : \mathbf{J} \to \mathbf{C}$ is a diagram, a **cocone** from $D$ is defined to be a natural transformation from $D$ to a constant functor. In summary, it is a pair $(L, \{\eta_\alpha\})$ with $L \in \text{ob}(\mathbf{C}), \eta_\alpha : D\alpha \to L$ with $\alpha \in \text{ob}(\mathbf{J})$, such that




is commutative for suitable morphisms $f$ in $\mathbf{J}$. This is dual to a cone.

**DEFINITION**

*Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram of type $\mathbf{J}$ in $\mathbf{C}$. A **colimit** of $D$ is defined to be a cocone $(L, \{\eta_\alpha\})$ from $D$ such that for every cocone $(B, \{\zeta_\alpha\})$ from $D$, there exists a unique morphism $\theta : L \to B$ such that $\zeta_\alpha = \theta\eta_\alpha$ for all $\alpha$.*

Once again, it is routine to show uniqueness up to isomorphism of this.

**EXAMPLES**

1. If $\mathbf{J}$ is a discrete category, then a cocone is simply a pair $(L, \{\eta_\alpha\})$ with $L \in \text{ob}(\mathbf{C})$ and $\eta_\alpha : D\alpha \to L$. The coherence diagram is automatic. It is easy to see that a colimit of $D$ is simply a coproduct of the $D\alpha$'s.
2. Let $\mathbf{J}$ be the category with only two objects $\alpha, \beta$ such that the only morphisms are the identity morphisms and two morphisms $\alpha \to \beta$. Then a colimit of a diagram is a difference cokernel [or coequalizer] of the two morphisms. This can be done with more than two morphisms as well.
3. Suppose $\mathbf{J}$ has three objects, $\alpha, \beta, \gamma$ and two nonidentity morphisms, $\gamma \to \alpha$ and $\gamma \to \beta$. Then a colimit of a diagram is a pushout.

What's quite unbelievable is that $\mathbf{V}(S)$ also contains all colimits! The material covered in the previous chapter can be used to prove this.

**THEOREM 2.4** *Colimits exist in $\mathbf{V}(S)$ for any diagram $D : \mathbf{J} \to \mathbf{V}(S)$.*

*Proof of Theorem 2.4.* Let $D : \mathbf{J} \to \mathbf{V}(S)$ be a diagram. Then let $A = \coprod_{\alpha \in \text{ob}(\mathbf{J})} D\alpha$, with injections $i_\alpha : D\alpha \to A$ for $\alpha \in \text{ob}(\mathbf{J})$. Now, let $\Theta$ be the congruence relation on $A$ generated by the following subset of $A \times A$:

$$\{(i_\beta D(f)(a), i_\alpha(a)) \mid f : \alpha \to \beta \text{ in } \mathbf{J}, a \in D\alpha\}$$

Set $L = A/\Theta$, $\pi : A \to L$ the canonical epimorphism and $\eta_\alpha = \pi i_\alpha$ for $\alpha \in \text{ob}(\mathbf{J})$. We claim that $(L, \{\eta_\alpha\})$ is a cocone from $D$. To show this, we need to show that $\eta_\beta D(f) = \eta_\alpha$ for $f : \alpha \to \beta$ in $\mathbf{J}$. This follows because for all $a \in D_\alpha$, $(i_\beta D(f)(a), i_\alpha(a)) \in \Theta$ by definition, so that $\eta_\beta D(f)(a) = \pi i_\beta D(f)(a) = \pi i_\alpha(a) = \eta_\alpha(a)$. Therefore, $\eta_\beta D(f) = \eta_\alpha$ and $(L, \{\eta_\alpha\})$ is a cocone.

Now suppose $(B, \{\zeta_\alpha\})$ is another cocone from $D$. Then since each $\zeta_\alpha : D\alpha \to B$ and $A$ is the coproduct of the $D\alpha$'s, there is a unique morphism

$\zeta : A \to B$ such that $\zeta i_\alpha = \zeta_\alpha$ for all $\alpha$. Whenever $f : \alpha \to \beta \in \mathbf{J}$ and $a \in D\alpha$,

$$\zeta i_\beta D(f)(a) = \zeta_\beta D(f)(a) = \zeta_\alpha(a) = \zeta i_\alpha(a)$$

because the $\zeta_\alpha$'s form a cocone; hence $(i_\beta D(f)(a), i_\alpha(a)) \in \ker\zeta$. Since the congruence relation $\Theta$ is generated by pairs of that form, $\Theta \subseteq \ker\zeta$, and $\zeta$ can be injectified [Theorem 1.10] to a morphism $\theta : L \to B$ satisfying $\zeta = \theta\pi$.

Furthermore, $\zeta_\alpha = \zeta i_\alpha = \theta\pi i_\alpha = \theta\eta_\alpha$ for all $\alpha$.

Since any homomorphism $\theta'$ satisfying $\zeta_\alpha = \theta'\eta_\alpha$ agrees with $\theta$ on all elements of images of the $\eta_\alpha$, but they generate $L$, $\theta$ is unique, completing the proof. ■

**EXERCISES**

1. Let $\mathbf{J}, \mathbf{C}$ be categories, and $\mathbf{C}^{\mathbf{J}}$ the functor category. Define the **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ by sending each $A \in \mathrm{ob}(\mathbf{C})$ to the constant functor onto $A$. For $f : A \to B$ in $\mathbf{C}$, $\Delta(f)$ is the natural transformation $\eta : \Delta A \Rightarrow \Delta B$ with $\eta_\alpha = f$ for all $\alpha$. Show that a limit of a diagram $D$ is a universal from $\Delta$ to the object $D$ of $\mathbf{C}^{\mathbf{J}}$, and that a colimit is a universal from $D$ to $\Delta$.

2. Suppose $\iota \in \mathrm{ob}(\mathbf{J})$ is a initial object. If $D : \mathbf{J} \to \mathbf{C}$ is a diagram, then $(D\iota, \{\eta_\alpha\})$ is a limit of $D$, where $\eta_\alpha$ is the result of applying $D$ to the unique morphism $\iota \to \alpha$ in $\mathbf{J}$. Dualize.

3. Show that any category with all products [including the terminal object] and equalizers has all limits as follows. Let $D : \mathbf{J} \to \mathbf{C}$ be a diagram. Now let $A = \prod_{\alpha \in \mathrm{ob}(\mathbf{J})} D\alpha$ and $P = \prod_{f \in \hom_{\mathbf{J}}(\alpha,\beta)} D\beta$, where the latter product is taken over all morphisms in $\mathbf{J}$. Denote the projections from $A$ as $p^1_\alpha : A \to D\alpha$ and the projections from $P$ as $p^2_f : P \to D\beta$, $f \in \hom(\alpha, \beta)$.

   (a) Show that there is a unique morphism $\varphi : A \to P$ such that $p^2_f\varphi = p^1_\beta$ for $f \in \hom(\alpha, \beta)$. [*Hint*: If you need a hint, think about how $P$ is defined.]

   (b) Show that there is also a unique $\psi : A \to P$ such that $p^2_f\psi = D(f)p^1_\alpha$ for $f \in \hom(\alpha, \beta)$.

   (c) Now let $\epsilon : L \to A$ be an equalizer of $\varphi$ and $\psi$; show that $(L, \{p^1_\alpha\epsilon\})$ is a limit of $D$.

   (d) In a variety $\mathbf{V}(S)$ in universal algebra, recall that products are direct products, and the equalizer of $f, g : A_1 \to A_2$ is the canonical monomorphism from the subalgebra $\{a \in A_1 \mid f(a) = g(a)\}$. Use this to find limits in $\mathbf{V}(S)$. Are they really different from Theorem 2.3?

4. Let $f_i : A_i \to B$ be morphisms in $\mathbf{C}$ for $i = 1, 2$. Then let $g_i : C \to A_i$, $i = 1, 2$ be a pullback of $f_1$ and $f_2$. Prove that if $f_1$ is monic then so is $g_2$.

5. A functor $F : \mathbf{C} \to \mathbf{D}$ is **continuous** if it *preserves limits*: Whenever $D : \mathbf{J} \to \mathbf{C}$ is a diagram and $(A, \{\eta_\alpha\})$ is a limit of $D$, then $(FA, \{F(\eta_\alpha)\})$ is a limit of $FD$. A **cocontinuous** functor is defined likewise, but for colimits.

   Let $T : \mathcal{V}(S_1) \to \mathcal{V}(S_2)$ be a takeoff of varieties.

   (a) The functor $F : \mathbf{V}(S_1) \to \mathbf{V}(S_2)$ given by Example 1 of Section 3, is continuous. [*Hint*: Theorem 2.3 shows how to construct the limit. What does the construction depend on?]

   (b) The functor $G : \mathbf{V}(S_2) \to \mathbf{V}(S_1)$ given by Example 12 of Section 3, is cocontinuous. [*Hint*: This is a variation of Exercise 14 of Section 1.11.]

   (c) If $\mathbf{C}$ is a complete category, then any functor $F : \mathbf{C} \to \mathbf{D}$ which preserves products [including the terminal object] and equalizers is continuous. Dualize.

# 2.7 - Hom Functors, Yoneda's Lemma and Representability

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

A hom functor is a functor $F : \mathbf{C} \to \mathbf{Set}$ defined in a special way. Though they have a seemingly basic definition, they are actually very important in the future sections of this chapter.

Fix $A \in \text{ob}(\mathbf{C})$, and define $F$ as follows:

1. For each $B \in \text{ob}(\mathbf{C})$, assign $FB$ to the set $\hom(A, B)$;

2. For each $f : B \to B'$ in $\mathbf{C}$, define $F(f)$ to be the set map from $\hom(A, B)$ to $\hom(A, B')$ sending $h \to fh$. [This set map is notated $\hom(A, f)$.]

It is feasibly shown that the data above defines a functor from $\mathbf{C}$ to $\mathbf{Set}$. This functor is denoted $\hom(A, -)$ and is called a **covariant hom functor**.

Now let $F : \mathbf{C} \to \mathbf{Set}$ be any functor and let $\eta$ be a natural transformation $\hom(A, -) \Rightarrow F$. Then for *any* object $B$ in $\mathbf{C}$, $\eta_B$ is a set map $\hom(A, B) \to FB$. In particular, taking $A$ for $B$, $\eta_A$ is a set map $\hom(A, A) \to FA$. Therefore, $a = \eta_A(1_A)$ is some element of $FA$. We claim that $a$ completely determines the natural transformation $\eta$. To see where $\eta_B$ sends $f : A \to B$, note the commutativity of

$$\begin{array}{ccc} \hom(A,A) & \xrightarrow{\eta_A} & FA \\ {\scriptstyle \hom(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \hom(A,B) & \xrightarrow{\eta_B} & FB \end{array}$$

due to $\eta$ being a natural transformation. Traveling $1_A$ along each pair of arrows yields $\eta_B(\hom(A, f)(1_A)) = \eta_B(f1_A) = \eta_B(f)$ and $F(f)(\eta_A(1_A)) = F(f)(a)$. Therefore, $\eta_B(f) = F(f)(a)$, which determines $\eta_B$.

It is straightforward to show that any $a$ is possible: if $a$ is an arbitrary element of $FA$, define $\eta_B : \hom(A, B) \to FB$ by $\eta_B(f) = F(f)(a)$ for each $B$. Then

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\eta_B} & FB \\ {\scriptstyle \hom(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \hom(A,B') & \xrightarrow{\eta_{B'}} & FB' \end{array}$$

is commutative for any $f : B \to B'$, because one direction yields $F(f)\eta_B$ and the other $\eta_{B'} \hom(A, f)$ and they are the same set map:

$F(f)(\eta_B(h)) = F(f)(F(h)(a)) = (F(f)F(h))(a) = F(fh)(a) = \eta_{B'}(fh) = \eta_{B'}(\hom(A, f)(h))$

Therefore, $\eta$ is a natural transformation $\hom(A, -) \Rightarrow F$. Also, $\eta_A(1_A) = F(1_A)(a) = 1_{FA}(a) = a$. What we have proved is summarized as follows.

**LEMMA 2.5** (YONEDA'S LEMMA) *Let* $\mathbf{C}$ *be a category,* $A \in ob(\mathbf{C})$*,* $F : \mathbf{C} \to \mathbf{Set}$ *be a functor. Then for every* $a \in FA$ *there is a unique natural transformation* $\eta : \hom(A, -) \Rightarrow F$ *such that* $a = \eta_A(1_A)$*. This natural transformation is given by* $\eta_B$ *being the map* $f \to F(f)(a)$ *from* $\hom(A, B) \to FB$ *for each* $B$*.*

Hopefully this sounds like Exercise 3 of Section 1.10 a bit.

Now let $F : \mathbf{V}(S) \to \mathbf{Set}$ be the forgetful functor. Then let $A$ be the free $\mathcal{V}(S)$-algebra given by a single symbol $a$, and let $\eta$ be the natural transformation $\hom(A, -) \Rightarrow F$ satisfying $\eta_A(1_A) = a$. We have $\eta_B(f) = F(f)(a)$ for each $f : A \to B$. Since $A$ is the free algebra given by $a$, then for every $b \in B$ there is a unique homomorphism $f : A \to B$ such that $f(a) = b$; this implies that $\eta_B$ is bijective, so that $\eta$ is actually a natural isomorphism. Hence $(A, a)$ is a special pair for the functor $F$, which yields the following definition.

**DEFINITION**

*Let* $F : \mathbf{C} \to \mathbf{Set}$ *be a functor. If there exists a pair* $(A, a)$ *with* $A \in ob(\mathbf{C}), a \in FA$ *such that the natural transformation* $\hom(A, -) \Rightarrow F$ *induced by Lemma 2.5 is a natural isomorphism,* $(A, a)$ *is called a* ***representative*** *of the functor* $F$*. If functor which has a representative is said to be* ***representable****.*

**EXAMPLES**

1. If $F : \mathbf{V}(S) \to \mathbf{Set}$ is the forgetful functor, we have seen that $F$ is representable with $(F_S(\Omega, \{a\}), a)$ as a representative.

2. Let $F : \mathbf{C} \to \mathbf{Set}$ be the constant functor onto the one-element set $\{a\}$. Then a representative of $F$ is $(I, a)$ with $I$ an initial object in $\mathbf{C}$. This is because for every $A \in \mathbf{C}$, $\hom(I, A)$ and $FA$ are both one-element sets, so they come in a unique natural bijection. Hence $F$ is representable if and only if $\mathbf{C}$ has an initial object; in particular, this holds if $\mathbf{C} = \mathbf{V}(S)$.

3. (Quotient algebras) Let $A$ be a $\mathcal{V}(S)$-algebra and $\Phi$ a congruence relation on $A$. Define $F : \mathbf{V}(S) \to \mathbf{Set}$ sending $B$ to the subset $\{h \in \hom(A, B) \mid \Phi \subseteq \ker h\}$ of $\hom(A, B)$. Then clearly for $f : B \to B'$, $\hom(A, f)$ sends elements of $FB$ to elements of $FB'$, so it can be restricted to a set map $F(f) : FB \to FB'$. It is easy to see that this data defines a functor.

Now let $\pi : A \to A/\Phi$ be the canonical epimorphism, then $\pi \in F(A/\Phi)$. We claim that $(A/\Phi, \pi)$ is a representative of $F$. This is because $\eta_B : \hom(A/\Phi, B) \Rightarrow FB$ sends $h \to F(h)(\pi) = h\pi$. The injectification theorem (1.10) shows that for all $f : A \to B$ such that $\Phi \subseteq \ker f$, there is a unique $h : A/\Phi \to B$ such that $f = h\pi$. This means $\eta_B$ is actually bijective, so $\eta$ is a natural isomorphism.

4. (Colimits) Let $D : \mathbf{J} \to \mathbf{C}$ be a fixed diagram of type $\mathbf{J}$ in $\mathbf{C}$. Define $F : \mathbf{C} \to \mathbf{Set}$ sending each object $B$ of $\mathbf{C}$ to the set of cocones $(B, \{\eta_\alpha\})$ from $D$ to $B$. If $f : B \to B'$ is a morphism in $\mathbf{C}$ and $(B, \{\eta_\alpha\})$ is a cocone from $D$, clearly $(B', \{f\eta_\alpha\})$ is also a cocone; this induces a set map $F(f) : FB \to FB'$. It is straightforward to show that $F$ is a functor.

Now let $(L, \{\eta_\alpha\})$ be a colimit of $D$. We claim that $L$, along with this cocone, is a representative of $F$. The natural transformation $\eta : \hom(L, -) \Rightarrow F$

assigns each $\eta_B, B \in \text{ob}(\mathbf{C})$ the map $\theta \to F(\theta)(L, \{\eta_\alpha\}) = (B, \{\theta\eta_\alpha\})$ from $\text{hom}(L, B) \to FB$. Recall that the virtue of being a colimit is that for any cocone $(B, \{\zeta_\alpha\})$, there is a unique $\theta : L \to B$ such that $\zeta_\alpha = \theta\eta_\alpha$ for every $\alpha$; that is, $(B, \{\zeta_\alpha\}) = \eta_B(\theta)$. Therefore $\eta_B$ is bijective.

Conversely, every representative of $F$ is a colimit of $D$; see Exercise 5 for proof.

We now fix $A \in \text{ob}(\mathbf{C})$ and define a contravariant functor as follows.

1. For each $B \in \text{ob}(\mathbf{C})$, assign $FB$ to the set $\text{hom}(B, A)$.

2. For each $f : B \to B'$ in $\mathbf{C}$, define $F(f)$ to be the map $\text{hom}(B', A) \to \text{hom}(B, A)$ sending $h \to hf$. [This set map is notated $\text{hom}(f, A)$; be wary that $B$ and $B'$ swap places.]

This functor is denoted $\text{hom}(-, A)$ and is called a **contravariant hom functor**. Another way to view this is that $\text{hom}_{\mathbf{C}}(-, A) = \text{hom}_{\mathbf{C}^{\text{op}}}(A, -)$ which is a covariant functor $\mathbf{C}^{\text{op}} \to \mathbf{Set}$. Yoneda's Lemma then dualizes to $\text{hom}(-, A)$ as follows:

**LEMMA 2.5** (CONTINUED) *Let $F : \mathbf{C} \to \mathbf{Set}$ be a* contravariant *functor, $A \in ob(\mathbf{C})$. Then for every $a \in FA$ there is a unique natural transformation $\eta : \text{hom}(-, A) \Rightarrow F$ such that $a = \eta_A(1_A)$. This natural transformation is given by $\eta_B$ being the map $f \to F(f)(a)$ from* $\text{hom}(B, A) \to FB$ *for each $B$.*

This immediately follows from the first one when $\mathbf{C}^{\text{op}}$ is used. When the pair $(A, a)$ is special enough for $\eta$ to be a natural isomorphism, $F$ is said to be **representable** with $(A, a)$ as a representative.

**EXAMPLES**

1. Let $F : \mathbf{C} \to \mathbf{Set}$ be the contravariant constant functor onto the one-element set $\{a\}$. Then a representative of $F$ is $(T, a)$ with $T$ a terminal object in $\mathbf{C}$. Hence $F$ is representable if and only if $\mathbf{C}$ has a terminal object.

2. (Subalgebras) Let $A$ be a $\mathcal{V}(S)$-algebra and $A'$ a subalgebra on $A$. Define $F : \mathbf{V}(S) \to \mathbf{Set}$ sending $B$ to the subset $\{h \in \text{hom}(B, A) \mid h(B) \subseteq A'\}$ of $\text{hom}(B, A)$. Then clearly for $f : B \to B'$, $\text{hom}(f, A)$ sends elements of $FB'$ to elements of $FB$, so it can be restricted to a set map $F(f) : FB' \to FB$. This data defines a contravariant functor.

Now let $\iota : A' \to A$ be the canonical monomorphism, then $\iota \in FA'$. We claim that $(A', \iota)$ is a representative of $F$. This is because $\eta_B : \text{hom}(B, A') \Rightarrow FB$ sends $h \to F(h)(\iota) = \iota h$. The bijectivity of each $\eta_B$ follows from the surjectification theorem (1.5).

3. (Limits) Let $D : \mathbf{J} \to \mathbf{C}$ be a fixed diagram of type $\mathbf{J}$ in $\mathbf{C}$. Define $F : \mathbf{C} \to \mathbf{Set}$ sending each object $B$ of $\mathbf{C}$ to the set of cones $(B, \{\eta_\alpha\})$ from $B$ to $D$. Then for $f : B' \to B$ a morphism in $\mathbf{C}$ and $(B, \{\eta_\alpha\})$ a cone to $D$, so is $(B', \{\eta_\alpha f\})$, inducing a set map $F(f) : FB \to FB'$. This defines a contravariant functor $F$. The reasoning above shows that a representative of $F$ is a limit of the diagram $D$.

4. Let $\mathcal{P}$ be the contravariant functor of Example 3 of Section 3. Now let $\mathbb{Z}_2$ be the set $\{0,1\}$, then $\{1\}$ is a subset of $\mathbb{Z}_2$, hence is in $\mathcal{P}(\mathbb{Z}_2)$. We claim that $\mathcal{P}$ is representable with $(\mathbb{Z}_2, \{1\})$ as a representative. The natural transformation $\eta : \hom(-, \mathbb{Z}_2) \Rightarrow \mathcal{P}$ sends a set $X$ to the map $\eta_X : \hom(X, \mathbb{Z}_2) \Rightarrow \mathcal{P}(X)$ sending $f \to F(f)(\{1\}) = f^{-1}(\{1\}) = \{x \in X \mid f(x) = 1\}$. Since for every subset $X'$ of $X$, the *characteristic function* $f : X \to \mathbb{Z}_2$ [$f(x) = 1$ for $x \in X'$, 0 for $x \in X - X'$] is the unique element of $\hom(X, \mathbb{Z}_2)$ sent to $X'$ by $\eta_X$, $\eta_X$ is bijective. Therefore, $\eta$ is a natural isomorphism.

The hom functors have a huge use in the next section.

**EXERCISES**

1. Use Yoneda's Lemma to show that for every $h : A' \to A$ in $\mathbf{C}$, there is a unique natural transformation $\eta : \hom(A, -) \Rightarrow \hom(A', -)$ such that $h = \eta_A(1_A)$.

2. Let $\mathbf{C}$ be an arbitrary category, and define $H : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ as follows. For each pair $(A, B)$ of objects of $\mathbf{C}$, $H(A, B) = \hom(A, B)$. To define $H(f)$ for $f : (A, B) \to (A', B')$ in $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$, note that $f$ is a pair of morphisms $f_1 : A' \to A, f_2 : B \to B'$ of $\mathbf{C}$. Assign $H(f)$ the set map $h \to f_2 h f_1$ from $\hom(A, B)$ to $\hom(A', B')$. Verify that this is a functor. It is called the **two-variable hom functor**.

3. Show that $\hom(A, -)$ is continuous. [See Exercise 5 of Section 6.] Conclude that every [covariant] representable functor is continuous.

4. Let $X$ be a set, $A \in \mathrm{ob}(\mathbf{C})$, and $A' = \coprod_{x \in X} A$ be a copower of $A$ with injections $i_x : A \to A'$, $x \in X$. Then define $u : X \to \hom(A, A')$ sending $x \to i_x$. Show that $(A', u)$ is a universal from $X$ to $\hom(A, -)$.

5. A representative of a functor $F$ is precisely a universal from the one-element set $\{\circ\}$ to $F$. Conclude that if $(A, a)$ and $(A', a')$ are two representatives of $F$, there is a unique isomorphism $\sigma : A \to A'$ such that $F(\sigma)(a) = a'$.

6. Let $f : A' \to A, g : B \to B'$ in $\mathbf{C}$. Explain why

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\hom(f,B)} & \hom(A',B) \\ {\scriptstyle \hom(A,g)}\downarrow & & \downarrow{\scriptstyle \hom(A',g)} \\ \hom(A,B') & \xrightarrow{\hom(f,B')} & \hom(A',B') \end{array}$$

is a commutative diagram. Use this to show that $F : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}}$ is a contravariant functor when one defines $FA = \hom(A, -)$ and for $f : A' \to A$ in $\mathbf{C}$, $F(f) : \hom(A, -) \Rightarrow \hom(A', -)$ is the natural transformation given by $\eta_B = \hom(f, B) : \hom(A, B) \to \hom(A', B)$.

# 2.8 - Adjunctions

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

We learned universals in Section 5. In this section, we use them to define a functor. Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor such that for every object $B$ in $\mathbf{D}$, there exists a universal $(U, u)$ from $B$ to $F$.

Now let $\hom(B, F-)$ be the functor $\hom(B, -)F : \mathbf{C} \to \mathbf{Set}$; it sends $A \to \hom_{\mathbf{D}}(B, FA)$ and a morphism $f : A \to A'$ to $\hom(B, F(f)) : \hom(B, FA) \to \hom(B, FA')$ [this makes sense since $F(f) : FA \to FA'$]. We claim that $\hom(B, F-)$ is representable with $(U, u)$ as a representative; to see this, define $\eta_A : \hom(U, A) \to \hom(B, FA)$ sending $h \to F(h)u$. Then clearly $\eta_U(1_U) = u$ and $\eta$ is a natural transformation from $\hom(U, -)$ to $\hom(B, F-)$; that is, the $\eta_A$'s are natural in the right variable. Also, by virtue of a universal, for every $A \in \mathrm{ob}(\mathbf{C}), f : B \to FA$ there is a unique morphism $h : U \to A$ such that $f = F(h)u$. This says that for every $A \in \mathrm{ob}(\mathbf{C})$, $\eta_A$ is a bijection. Therefore, $\eta$ is a natural isomorphism.

We proceed to define a functor $G : \mathbf{D} \to \mathbf{C}$ as in Example 12 of Section 3. For each $B \in \mathbf{D}$, let $(GB, u_B)$ be any universal from $B$ to $F$. then the $\eta_A$'s in the above paragraph are bijections $\hom(GB, A) \to \hom(B, FA)$ which are natural in the right variable. To define $G(f)$ for $f : B \to B'$ in $\mathbf{D}$, note that by the universality of $(GB, u_B)$ there is a unique morphism $\tilde{f} : GB \to GB'$ such that $F(\tilde{f})u_B = u_{B'}f$:

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ {\scriptstyle u_B}\downarrow & & \downarrow{\scriptstyle u_{B'}} \\ FGB & \xrightarrow{F(\tilde{f})} & FGB' \end{array}$$

Set $G(f) = \tilde{f}$. Then $FG(f)u_B = u_{B'}f$ for all $f : B \to B'$ in $\mathbf{D}$. For $f : B \to B', g : B' \to B''$ in $\mathbf{D}$, the commutativity of the squares in

$$\begin{array}{ccccc} B & \xrightarrow{f} & B' & \xrightarrow{g} & B'' \\ {\scriptstyle u_B}\downarrow & & \downarrow{\scriptstyle u_{B'}} & & \downarrow{\scriptstyle u_{B''}} \\ FGB & \xrightarrow{FG(f)} & FGB' & \xrightarrow{FG(g)} & FGB'' \end{array}$$

and the functorial property $FG(g)FG(f) = F(G(g)G(f))$ imply that this diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{gf} & B'' \\ {\scriptstyle u_B}\downarrow & & \downarrow{\scriptstyle u_{B''}} \\ FGB & \xrightarrow{F(G(g)G(f))} & FGB'' \end{array}$$

Therefore, $G(g)G(f)$ satisfies the property satisfied by only the morphism $G(gf)$, meaning $G(gf) = G(g)G(f)$. Likewise, $G(1_B) = 1_{GB}$. Therefore, $G$ is a functor.

Now for all $A \in \text{ob}(\mathbf{C}), B \in \text{ob}(\mathbf{D})$, define $\eta_{B,A} : \hom(GB, A) \to \hom(B, FA)$ as above; $\eta_{B,A}(h) = F(h)u_B$. Then we have already seen that for each fixed $B$, $A \to \eta_{B,A}$ is a natural isomorphism from $\hom(GB, -) \Rightarrow \hom(B, F-)$. We claim that the $\eta$'s are natural in the left variable now; that is, for each fixed $A$, $B \to \eta_{B,A}$ is a natural isomorphism from $\hom(G-, A) \Rightarrow \hom(-, FA)$. Well, suppose $f : B' \to B$ in $\mathbf{D}$, then

$$\begin{array}{ccc} \hom(GB, A) & \xrightarrow{\eta_{B,A}} & \hom(B, FA) \\ {\scriptstyle \hom(G(f),A)}\downarrow & & \downarrow{\scriptstyle \hom(f,FA)} \\ \hom(GB', A) & \xrightarrow{\eta_{B',A}} & \hom(B', FA) \end{array}$$

is commutative, because for all $h \in \hom(GB, A)$,

$$\hom(f, FA)(\eta_{B,A}(h)) = \eta_{B,A}(h)f = F(h)u_B f$$

$$\eta_{B',A}(\hom(G(f), A)(h)) = \eta_{B',A}(hG(f)) = F(hG(f))u_{B'} = F(h)FG(f)u_{B'}$$

and they are equal because $FG(f)u_{B'} = u_B f$. Thus $\eta$ is natural in both variables separately. This leads to the following definition.

**DEFINITION**

*An* ***adjunction*** *a triple* $(G, F, \eta)$ *where* $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ *are functors and* $\eta$ *assigns each* $B \in ob(\mathbf{D}), A \in ob(\mathbf{C})$ *a bijection* $\eta_{B,A} : \hom(GB, A) \to \hom(B, FA)$ *and is natural in both variables as above. In this case,* $\eta$ *is called the* ***adjugant****,* $G$ *the* ***left adjoint functor*** *and* $F$ *the* ***right adjoint functor****.*

We have shown that the functor which sends objects in the codomain category to their universals — in particular, the functor in Example 12 of Section 3 — is a left adjoint functor of $F$. Surprisingly, the converse is true as well:

**THEOREM 2.6** *Let* $(G, F, \eta)$ *be an adjunction, wth* $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$*. Then for every* $B \in ob(\mathbf{D})$, $(GB, \eta_{B,GB}(1_{GB}))$ *is a universal from* $B$ *to* $F$*.*

*Proof of Theorem 2.6.* Let $u = \eta_{B,GB}(1_{GB}) : B \to FGB$. Then suppose $A \in \text{ob}(\mathbf{C})$ and $f : B \to FA$. We wish to show that there is a unique morphism $h : GB \to A$ such that

$$\begin{array}{ccc} B & \xrightarrow{u} & FGB \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F(h)} \\ & & FA \end{array}$$

is commutative. To show this, we need only show that $\eta_{B,A}$ is precisely the map $h \to F(h)u$ from $\hom(GB, A) \to \hom(B, FA)$, for then the bijectivity of $\eta_{B,A}$ implies that there is a unique $h$ such that $f = F(h)u$, proving the theorem.

For each $h : GB \to A$, the following diagram

$$\begin{array}{ccc} \hom(GB, GB) & \xrightarrow{\eta_{B,GB}} & \hom(B, FGB) \\ {\scriptstyle \hom(GB,h)}\downarrow & & \downarrow{\scriptstyle \hom(B,F(h))} \\ \hom(GB, A) & \xrightarrow{\eta_{B,A}} & \hom(B, FA) \end{array}$$

commutes due to the $\eta$'s being natural in the right variable. In particular, sending $1_{GB}$ along each pair of arrows yields:

$$\eta_{B,A}(\hom(GB, h)(1_{GB})) = \eta_{B,A}(h1_{GB}) = \eta_{B,A}(h)$$

$$\hom(B, F(h))(\eta_{B,GB}(1_{GB})) = \hom(B, F(h))(u) = F(h)u$$

Therefore, $\eta_{B,A}(h) = F(h)u$, and $\eta_{B,A}$ sends each $h$ to $F(h)u$. ■

The fact that a left adjoint functor necessarily sends objects to their universals, makes them unique up to a natural isomorphism.

**THEOREM 2.7** *Any two left adjoint functors of $F$ are naturally isomorphic.*

*Proof of Theorem 2.7.* Let $(G, F, \eta)$ and $(G', F, \zeta)$ be adjunctions with $F : \mathbf{C} \to \mathbf{D}$, $G, G' : \mathbf{D} \to \mathbf{C}$. For $B \in \text{ob}(\mathbf{D})$ assign $u_B = \eta_{B,GB}(1_{GB})$ and $v_B = \zeta_{B,G'B}(1_{G'B})$. Then by Theorem 2.6, $(GB, u_B)$ and $(G'B, v_B)$ are universals from $B$ to $F$. Therefore, there is a unique isomorphism $\sigma_B : GB \to G'B$ such that $v_B = F(\sigma_B)u_B$. We claim that $\sigma$ is a natural isomorphism $G \Rightarrow G'$ when defined this way. To show this, we must show that for all $f : B \to B'$,

$$\begin{array}{ccc} GB & \xrightarrow{\sigma_B} & G'B \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle G'(f)} \\ GB' & \xrightarrow{\sigma_{B'}} & G'B' \end{array}$$

is commutative. Applying $\eta_{B,G'B'}$ to both composite arrows,

$$\eta_{B,G'B'}(G'(f)\sigma_B) = F(G'(f)\sigma_B)u_B = FG'(f)F(\sigma_B)u_B = FG'(f)v_B = v_{B'}f$$

$$\eta_{B,G'B'}(\sigma_{B'}G(f)) = F(\sigma_{B'}G(f))u_B = F(\sigma_{B'})FG(f)u_B = F(\sigma_{B'})u_{B'}f = v_{B'}f$$

Whence $G'(f)\sigma_B = \sigma_{B'}G(f)$ because $\eta_{B,G'B'}$ is a bijection. Therefore, $\sigma$ is a natural isomorphism, and $G$ and $G'$ are naturally isomorphic. ■

Everything we have shown dualizes to universals from a functor to an object. A functor sending objects to those kinds of universals is a right adjoint functor. We leave the verification of this to the reader.

**EXERCISES**

1. Describe the left adjoint functor of:

   (a) The identity functor on any category.

   (b) A takeoff of varieties.

   (c) The functor **Set** $\to$ **Set** sending every set $X$ to $X^Y$, with $Y$ a fixed set. [*Hint*: send each $X$ to $Y \times X$.]

   (d) The functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ in Exercise 1 of Section 6.

   (e) The constant functor onto some terminal object in **D**.

2. Show that for a fixed monoid $M$, the forgetful functor $M-$**act** $\to$ **Set** has both a left adjoint and a right adjoint.

3. Let $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ be functors. Show that the following two are functors from $\mathbf{D}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$.

   (a) $\hom(G-, -)$, sending $(B, A) \to \hom(GB, A)$ and $(g, f)$ with $g : B \to B'$ [that is, $g : B' \to B$ in **D**] and $f : A \to A'$ the map $\hom(GB, A) \to \hom(GB', A')$ sending $h \to fhG(g)$.

   (b) $\hom(-, F-)$, sending $(B, A) \to \hom(B, FA)$ and $(g, f)$ with $g : B \to B'$ and $f : A \to A'$ the map $\hom(B, FA) \to \hom(B', FA')$ sending $h \to F(f)hg$.

   (c) An adjunction is [identifiably] a triple $(G, F, \eta)$ with $\eta$ a natural isomorphism $\hom(G-, -) \Rightarrow \hom(-, F-)$.

4. Suppose $(G_1, F_1, \eta^1)$ and $(G_2, F_2, \eta^2)$ are adjunctions with

$$F_1 : \mathbf{C} \to \mathbf{D}, G_1 : \mathbf{D} \to \mathbf{C}$$

$$F_2 : \mathbf{D} \to \mathbf{E}, G_2 : \mathbf{E} \to \mathbf{D}$$

   For $B \in \mathbf{E}, A \in \mathbf{C}$, set $\eta_{B,A} = (\eta^2_{B,F_1A})(\eta^1_{G_2B,A})$. Show that $(G_1G_2, F_2F_1, \eta)$ is an adjunction.

5. Let $(G, F, \eta)$ be an adjunction.

   (a) For each $B \in \mathrm{ob}(\mathbf{D})$, assign $\delta_B = \eta_{B,GB}(1_{GB}) : B \to FGB$. Then $\delta$ is a natural transformation $1_{\mathbf{D}} \Rightarrow FG$. $\delta$ is called the **unit** of the adjunction.

   (b) For each $A \in \mathrm{ob}(\mathbf{C})$, assign $\epsilon_A = \eta^{-1}_{FA,A}(1_{FA}) : GFA \to A$. Then $\epsilon$ is a natural transformation $GF \Rightarrow 1_{\mathbf{C}}$. $\epsilon$ is called the **counit** of the adjunction.

   (c) $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$, using the notation from Exercise 8 of Section 3. These are summarized as the following being the identity natural transformations:

$$F \xrightarrow{\delta F} FGF \xrightarrow{F\epsilon} F$$

$$G \xrightarrow{G\delta} GFG \xrightarrow{\epsilon G} G$$

(d) Suppose $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ are any functors, and $\delta : 1_{\mathbf{D}} \Rightarrow FG$ and $\epsilon : GF \Rightarrow 1_{\mathbf{C}}$ are any natural transformations, such that $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$. Show that there is a unique adjugant $\eta$ making $(G, F, \eta)$ an adjunction with unit $\delta$ and counit $\epsilon$. Thus adjunctions could be defined in terms of their units and counits.

6. Every right adjoint functor is continuous, and every left adjoint functor is cocontinuous.

7. Give necessary and sufficient conditions on an object $A$ in a category $\mathbf{C}$ for $\hom(A, -)$ to have a left adjoint functor. [*Hint*: Exercise 4 of Section 7.]

8. Suppose $(G, F, \eta)$ is an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit $\delta$ and counit $\epsilon$. Then suppose $(G', F', \eta')$ is another adjunction with $F' : \mathbf{C}' \to \mathbf{D}', G' : \mathbf{D}' \to \mathbf{C}'$, unit $\delta'$ and counit $\epsilon'$.

   A **morphism** from $(G, F, \eta) \to (G', F', \eta')$ is defined to be pair $(K, L)$ with $K : \mathbf{C} \to \mathbf{C}'$ and $L : \mathbf{D} \to \mathbf{D}'$ functors such that $LF = F'K$, $KG = G'L$ and $L\delta = \delta' L$. [The last condition makes sense because $L\delta : L \Rightarrow LFG$ and $\delta' L : L \Rightarrow F'G'L$, yet the first two conditions imply $LFG = F'KG = F'G'L$.]

   (a) Given the conditions $LF = F'K$ and $KG = G'L$, show that the following three statements are equivalent and hence any of them could be the third condition: $L\delta = \delta' L$; $K\epsilon = \epsilon' K$; the diagram

$$\begin{array}{ccc} \hom(GB, A) & \xrightarrow{\eta_{B,A}} & \hom(B, FA) \\ {\scriptstyle K_{GB,A}}\downarrow & & \downarrow{\scriptstyle L_{B,FA}} \\ \hom(G'LB, KA) & \xrightarrow{\eta'_{LB,KA}} & \hom(LB, F'KA) \end{array}$$

   commutes for all $A \in \mathbf{C}, B \in \mathbf{D}$. [By $K_{GB,A}$, of course, I mean the set map $f \to K(f)$ from $\hom(GB, A) \to \hom(KGB, KA) = \hom(G'LB, KA)$.]

   (b) If $(K, L)$ is a morphism $(G, F, \eta) \to (G', F', \eta')$ and $(K', L')$ a morphism $(G', F', \eta') \to (G'', F'', \eta'')$, then $(K'K, L'L)$ is a morphism $(G, F\eta) \to (G'', F'', \eta'')$. This induces a composition $(K', L') \circ (K, L) = (K'K, L'L)$.

   (c) $(1_{\mathbf{C}}, 1_{\mathbf{D}})$ is a morphism $(G, F, \eta) \to (G, F, \eta)$. Define $1_{(G,F,\eta)} = (1_{\mathbf{C}}, 1_{\mathbf{D}})$.

   (d) Let $\mathcal{C}$ be a class of categories. Show that $\mathbf{Adj}(\mathcal{C})$ is a category defined as follows: objects are adjunctions between categories in $\mathcal{C}$, and morphisms are morphisms of adjunctions.

   (e) Establish two canonical functors from $\mathbf{Adj}(\mathcal{C})$ to the category of categories in $\mathcal{C}$ with functors as morphisms.

# 2.9 - Concrete Categories

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

**This section is not a prerequisite of any other and may be skipped if desired.**

Recall that the category $\mathbf{V}(S)$ has a forgetful functor $\mathbf{V}(S) \to \mathbf{Set}$, which is faithful and continuous. This gives better definitions of "injective," "surjective," "subobject," and "quotient object." One must be cautious though, about the fact that objects of $\mathbf{V}(S)$ are *not* identifiably sets. They do have underlying sets though, and the morphisms are in effect functions of the sets. The functor nature also implies that composition of morphisms and identity morphisms agree with those of the set maps. These kinds of categories have a special name.

**DEFINITION**

*A* ***concrete category*** *is a pair* $(\mathbf{C}, F)$ *with* $\mathbf{C}$ *a category and* $F : \mathbf{C} \to \mathbf{Set}$ *a faithful covariant functor.* $F$ *is called the* ***forgetful functor*** *for* $\mathbf{C}$*, and for each* $A \in \mathbf{C}$*,* $FA$ *is called the* ***underlying set*** *[or* ***carrier****] of* $A$*.*

Since $F$ is faithful, for $f : A \to B$ in $\mathbf{C}$ the map $f \to F(f)$ from $\hom(A, B) \to \hom(FA, FB)$ is injective. Hence $\hom(A, B)$ can be viewed as a subset of $\hom(FA, FB) = FB^{FA}$. One can therefore say a set map $FA \to FB$ is *admitted* as a morphism $A \to B$ or not.

**EXAMPLES**

1. $\mathbf{V}(S)$ is a concrete category. It has limits for all diagrams which match up with $\mathbf{Set}$'s limits because the forgetful functor is continuous. It also has colimits for all diagrams, but those don't match up with the colimits in $\mathbf{Set}$.

2. Since there are faithful functors $\mathbf{V}(S)-\mathbf{sub}, \mathbf{V}(S)-\mathbf{con} \to \mathbf{V}(S)$ [see Exercise 7 of Section 1], composing them with the forgetful functor $\mathbf{V}(S) \to \mathbf{Set}$ makes $\mathbf{V}(S)-\mathbf{sub}$ and $\mathbf{V}(S)-\mathbf{con}$ concrete categories. This can be generalized; if $\mathbf{C}$ is a concrete category and $F : \mathbf{D} \to \mathbf{C}$ is a faithful functor, $\mathbf{D}$ becomes a concrete category.

3. Consider the functor $S : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ sending $(X, Y) \to X \uplus Y$ and $(f, g)$ for $f : X \to X', g : Y \to Y'$ the map $f \uplus g : X \uplus Y \to X' \uplus Y'$ which sends $x \in X$ to $f(x)$ and $y \in Y$ to $g(y)$. It can be shown that $S$ is faithful [this is left to the reader]. This makes $\mathbf{Set} \times \mathbf{Set}$ a concrete category.

4. With parts 2 and 3 combined, the product of any two concrete categories is a concrete category [though one must define it carefully].

If $\mathbb{C}$ is a concrete category with forgetful functor $F : \mathbf{C} \to \mathbf{Set}$ and $X$ is a set, a universal from $X$ to $F$ is called a **free object** given by the set $X$. If every set $X$ has a free object, then the left adjoint functor of $F$ is called the **free-object giving functor** from $\mathbf{Set} \to \mathbf{C}$.

For example, for $\mathbf{V}(S)$ every set has a free object. But this fails for the functor $S$ in Example 3 [Exercise 4].

## Subobjects, quotient objects and Cartesian products

For a concrete category $(\mathbf{C}, F)$, one can conveniently define a morphism $f$ in $\mathbf{C}$ **injective** [**surjective**] if $F(f)$ is injective [surjective]. Then using the fact that $F$ is faithful, it follows that injective morphisms are monic and surjective morphisms are epic. Now call a morphism $f$ **bijective** if it is both injective and surjective. Then all isomorphisms are clearly bijective, but not conversely: a bijective morphism $A \to B$ in $\mathbf{C}$ is only an isomorphism if its inverse is admitted as a morphism $B \to A$.

We now treat objects in concrete categories set theoretically, and introduce the concept of an subobject / quotient object:

**DEFINITION**

*Let $\mathbf{C}$ be a concrete category, $A, B$ objects in $\mathbf{C}$. An injective morphism $\iota : B \to A$ is called an **embedding** provided that whenever $f : C \to A$ is a morphism such that $f(C) \subseteq \iota(B)$, the surjectified result $f_1$ satisfying $f = \iota f_1$ is admitted as a morphism $C \to B$. In this case, $(B, \iota)$ is called a **subobject** of $A$.*

*A surjective morphism $\pi : A \to B$ is called a **quotient map** provided that whenever $f : A \to C$ is a morphism such that $\ker \pi \subseteq \ker f$, the injectified result $\overline{f}$ satisfying $f = \overline{f}\pi$ is admitted as a morphism $B \to C$. In this case, $(B, \pi)$ is called a **quotient object** of $A$.*

At this point, it is convenient to introduce the notion of a **structure-based** concrete category. In such a category, elements of an object's set can be relabeled in any way to get a deterministic result.

**DEFINITION**

*Let $(\mathbf{C}, F)$ be a concrete category. Then $(\mathbf{C}, F)$ is **structure-based** if for any $A \in ob(\mathbf{C})$, set $X$ and bijection $\sigma : FA \to X$, there is a unique object $A' \in ob(\mathbf{C})$ such that $FA' = X$ and $\sigma$ is an isomorphism in $\hom(A, A')$.*

The definition can be rephrased as follows:

1. For any bijection $\sigma : FA \to X$, there is an object $A' \in \text{ob}(\mathbf{C})$ with $FA' = X$ and $\sigma$ admitted as an isomorphism in $\hom(A, A')$;
2. Whenever $A, A' \in \text{ob}(\mathbf{C})$ with $FA = FA'$ and $1_A$ is admitted as an isomorphism in $\hom(A, A')$, then $A$ and $A'$ are the same object.

This is because the second of those conditions states the uniqueness of the object $A'$. Every concrete category we have mentioned so far is structure-based. Can you come up with an example of a concrete category that isn't structure-based?

Thus in a *structure-based* concrete category, one can define $B$ to be a subobject of $A$ [notation $B \subseteq A$] if $FB \subseteq FA$ and the inclusion map $FB \hookrightarrow FA$ is in $\hom(B, A)$ as an embedding. Any subobject in the sense of the previous definition turns into one of these. In the next section we deal with only structure-based concrete categories, and we shall stick to this definition of a subobject.

Likewise if $\pi : A \to B$ is a quotient map, the codomain can be changed into a unique object such that its underlying set is the quotient set $A/\ker\pi$ and the morphism is the canonical epimorphism. We shall form quotient objects by taking the quotient set from this point.

**DEFINITION**

*Let $(\mathbf{C}, F)$ be a structure-based concrete category, $A_\alpha \in ob(\mathbf{C})$. Then a **(Cartesian) product** of the $A_\alpha$'s is an object $A$ such that:*

*1. $FA = \prod FA_\alpha$.*

*2. The projection maps $p_\alpha : \prod FA_\alpha \to FA_\alpha$ are admitted in* $\hom(A, A_\alpha)$.

*3. Whenever $B$ is an object in $ob(\mathbf{C})$ and $f_\alpha : B \to A_\alpha$, the coordinate map $f : FB \to \prod FA_\alpha$ satisfying $p_\alpha f = f_\alpha$ is admitted in* $\hom(B, A)$.

It is clear that a Cartesian product of objects is a product in the categorical sense. It follows that since $\mathbf{C}$ is structure-based, the object $A$ is unique, and can be referred to as *the* Cartesian product of the $A_\alpha$'s.

The striking controversy is that "product" in a concrete category could refer to either the categorial product or the Cartesian product. In the next section, it will always mean the latter.

**EXERCISES**

1. Let $(\mathbf{C}, F)$ and $(\mathbf{C}', F')$ be concrete categories. Define a **takeoff** $\mathbf{C} \to \mathbf{C}'$ to be a functor $T : \mathbf{C} \to \mathbf{C}'$ such that $F'T = F$.

   (a) Takeoffs are always faithful.

   (b) The concrete categories with takeoffs as morphisms form a category. [Assume hom-sets are allowed to be proper classes in this occasion.]

   (c) A takeoff of varieties coincides with a takeoff of the concrete categories.

   (d) Informally, what can you say about takeoffs?

2. Let $P$ be the functor $\mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ sending $(X, Y) \to X \times Y$. [See Exercise 7 of Section 4.] Is $P$ faithful? [*Hint*: Consider where $P$ sends $\hom((\mathbb{Z}, \emptyset), (\mathbb{Z}, \emptyset))$.]

3. Define a **congruence relation** $\Phi$ on a category $\mathbf{C}$ as follows:

   (1) For each $A, B \in \mathrm{ob}(\mathbf{C})$, an equivalence relation $\Phi_{A,B}$ on $\hom(A, B)$ is equipped.

   (2) Whenever $f, f' \in \hom(A, B)$, $g, g' \in \hom(B, C)$, $f\Phi_{A,B}f'$ and $g\Phi_{B,C}g'$, then $gf\Phi_{A,C}g'f'$.

   (a) Define a new category $\mathbf{C}/\Phi$ by assigning $\mathrm{ob}(\mathbf{C}/\Phi) = \mathrm{ob}(\mathbf{C})$ and $\hom_{\mathbf{C}/\Phi}(A, B) = \hom_{\mathbf{C}}(A, B)/\Phi_{A,B}$. For $\overline{f} \in \hom_{\mathbf{C}/\Phi}(A, B)$, $\overline{g} \in \hom_{\mathbf{C}/\Phi}(B, C)$, set $\overline{g}\overline{f} = \overline{gf}$. Then set $1_A = \overline{1_A} \in \hom_{\mathbf{C}/\Phi}(A, A)$. Explain why this is well-defined, and show that $\mathbf{C}/\Phi$ is a category. It is called the **quotient category** of $\mathbf{C}$ by $\Phi$.

(b) $\pi : \mathbf{C} \to \mathbf{C}/\Phi$ defined by $\pi A = A$, $\pi(f) = \overline{f}$ is a full functor which is bijective on the objects.

(c) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor [assumed covariant]. Define $\Theta_{A,B}$ to be the relation $\{(f, g) \in \hom(A, B)^2 \mid F(f) = F(g)\}$; show that $\Theta$ is a congruence relation on $\mathbf{C}$ when defined that way.

(d) If $\pi : \mathbf{C} \to \mathbf{C}/\Theta$ is defined as in part (b), there is a unique faithful functor $\overline{F} : \mathbf{C}/\Theta \to \mathbf{D}$ such that $F = \overline{F}\pi$. [*Hint*: This is similar to Theorem 1.10 in Chapter 1.]

(e) Now explain how to make $\mathbf{C}$ a concrete category given *any* functor $\mathbf{C} \to \mathbf{Set}$.

4. Let $S$ be the functor in Example 3, sending a pair of sets to its disjoint union, and making $\mathbf{Set} \times \mathbf{Set}$ a concrete category. If $X$ is a set, free object given by $X$ would be a pair of sets $(A, B)$ and a function $i : X \to A \uplus B$ such that whenever $f : X \to A' \uplus B'$ is a function, there are unique functions $a : A \to A', b : B \to B'$ such that $f = (a \uplus b)i$. For $X = \emptyset$, obviously $(\emptyset, \emptyset)$ works. However, if $X \neq \emptyset$ show that no such pair of sets exists. [*Hint*: Pick an element of $X$ and consider what parts of the disjoint unions $i$ and $f$ send them to.]

5. If a structure-based concrete category contains at least one nonempty set as an object, then it is not small.

# 2.10 - Algebraic Categories

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

**This section is not a prerequisite of any other and may be skipped if desired.**

*Note*: In this section, if $A$ is an object of a structure-based concrete category and $B$ is a subset of $A$'s set, then $B$ is said to be *admitted as a subobject* of $A$ if there exists an object whose set is $B$ such that $\hom(B, A)$ contains the inclusion map $B \hookrightarrow A$ as an embedding. This object is seen to be unique due to the category being structure-based. Likewise, if $\Phi$ is an equivalence relation on $A$, $A/\Phi$ is said to be *admitted as a quotient object* of $A$ if there exists an object whose set is $A/\Phi$ such that $\hom(A, A/\Phi)$ admits the quotient map $A \to A/\Phi$.

Recall from the previous section that that $\mathbf{V}(S)$ is a structure-based concrete category. However, it is more than just that, and we shall find a somewhat nonconstructive description of $\mathbf{V}(S)$. Some properties of $\mathbf{V}(S)$ that don't hold in an arbitrary structure-based concrete category are:

1. For every set there is a free object.
2. Products exist for any batch of objects.
3. If $f : A \to B$ is any morphism, the category admits the quotient object $A/\ker f$, the subobject $f(A)$ of $B$ and an isomorphism $\sigma : A/\ker f \to B$ such that

$$A \xrightarrow{\pi} A/\ker f \xrightarrow{\sigma} f(A) \xrightarrow{\iota} B$$

is equal to $f$, with $\pi$ and $\iota$ the canonical maps.

4. If $\Phi \subseteq A \times A$ is an equivalence relation on $A$ which is a subobject of $A \times A$, then the quotient set $A/\Phi$ is admitted as a quotient object of $A$.
5. (Finitary axiom) If $\{A_\alpha\}$ is a batch of subobjects of $A$ which is **directed**, [meaning for all $A_\alpha, A_\beta$ in the batch there exists $A_\gamma$ in the batch such that $A_\alpha \subseteq A_\gamma$ and $A_\beta \subseteq A_\gamma$], the union $\bigcup A_\alpha$ is a subobject.

A structure-based concrete category $\mathbf{C}$ satisfying rules 1-5 above is called a **finitary algebraic category**. If $\mathbf{C}$ satisfies rules 1-4 but not necessarily 5, $\mathbf{C}$ is just called an **algebraic category**. See Exercise 1 for an example.

Thus $\mathbf{V}(S)$ is a finitary algebraic category. In this section we show that every finitary algebraic category is of this form.

**THEOREM 2.8** *Every finitary algebraic category* $\mathbf{C}$ *is the category* $\mathbf{V}(S)$ *given by some variety* $\mathcal{V}(S)$ *in universal algebra.*

*Proof of Theorem 2.8* Let $\mathbf{C}$ be a finitary algebraic category. First note that property 3 implies that in $\mathbf{C}$, all injective morphisms are embeddings, and all surjective morphisms are quotient maps.

We proceed to define a signature $\Omega$ by letting $\Omega(n)$, for each $n \geq 0$, be the set $F_n$ underlying the free object given by the $n$-element set $I_n = \{x_1, x_2, \ldots, x_n\}$. Identify $x_1, x_2, \ldots, x_n$ with their images in the injection map $I_n \to F_n$. We then give each object $A$ an $\Omega$-algebra structure as follows. For each $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in A$, $\omega$ is an element of $F_n$. Define $(\omega a_1 a_2 \ldots a_n)$ to be $f(\omega)$ where $f$ is the unique morphism $F_n \to A$ extending the set map $x_i \to a_i$ from $I_n \to A$. The reader should take some time to understand the case when $n = 0$.

Now each object of $\mathbf{C}$ is an $\Omega$-algebra. We must prove two things to complete the proof:

(1) The morphisms between two objects of $\mathbf{C}$ are precisely the homomorphisms [i.e., operation-preserving maps] between the algebras.

(2) The class of $\Omega$-algebras is a variety.

It will follow that $\mathbf{C}$ is the category given by a variety.

To prove (1), let $A, B \in \mathbf{C}$. Suppose $f : A \to B$ is a morphism in $\mathbf{C}$. Then take $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in A$. We must show that $f(\omega a_1 a_2 \ldots a_n) = (\omega f(a_1) f(a_2) \ldots f(a_n))$. Let $\psi_A$ be the map $x_i \to a_i$ from $I_n \to A$, and $\psi_B$ be the map $x_i \to f(a_i)$ from $I_n \to B$. Then $\psi_A$ and $\psi_B$ extend to unique morphisms $\varphi_A : F_n \to A, \varphi_B : F_n \to B$. By definition of the $\Omega$-structures on the objects in $\mathbf{C}$,

$$\varphi_A(\omega) = (\omega a_1 a_2 \ldots a_n)$$

$$\varphi_B(\omega) = (\omega f(a_1) f(a_2) \ldots f(a_n))$$

Now, $\varphi_B$ is the *unique* morphism $F_n \to B$ sending each $x_i \to f(a_i)$. It is easy to see that $f\varphi_A$ also satisfies this, whence $f\varphi_A = \varphi_B$ by uniqueness. Applying these equal morphisms to $\omega$ [the element of $F_n$] yields the desired statement.

Conversely, suppose $f : A \to B$ is a homomorphism of the $\Omega$-algebras. Now let $F_A$ be the free object given by $A$, then the set map $f$ extends to a unique morphism $\eta : F_A \to B$ in $\mathbf{C}$, and the identity map $A \to A$ extends to a unique morphism $\epsilon : F_A \to A$. Evidently $\epsilon$ is surjective because it is the retraction of a set map. We claim that $\eta = f\epsilon : F_A \to B$. From that it will follow that $\ker \epsilon \subseteq \ker \eta$, and therefore, since $\epsilon$ is a quotient map [see the first paragraph of the proof], the injectified result, $f$, is admitted in $\mathbf{C}$.

Take any $a \in F_A$. By Exercise 4(c), $a \in F_{A'}$ for some finite subset $A'$ of $A$. Label the elements of $A'$ as $\{a_1, a_2, \ldots, a_n\}$, where $n = |A'|$. Then there is an bijection $F_n \to F_{A'}$ sending $x_i \to a_i$, and $a$ gets mapped to by some $\omega \in F_n = \Omega(n)$. By definition $a = (\omega a_1 a_2 \ldots a_n)$ [the expression in $F_A$, not its evaluation in $A$]. Since $\eta$ sends elements of $A$ to where $f$ sends them,

$$\eta(a) = (\omega f(a_1) f(a_2) \ldots f(a_n))$$

$$\epsilon(a) = (\omega a_1 a_2 \ldots a_n)$$

where the latter expression is the actual evaluation in the algebra $A$, and the former is the evaluation in the algebra $B$. The hypothesized statement $f(\omega a_1 a_2 \ldots a_n) = (\omega f(a_1) f(a_2) \ldots f(a_n))$ implies that $\eta(a) = f\epsilon(a)$. This holds for all $a \in F_A$, thus $\eta = f\epsilon$.

Now that we have (1) proved, we must show (2). To show that the class is a variety, by Theorem 1.26 it suffices to show that it is closed under subalgebras, homomorphic images and products, and contains $T(\Omega)$.

To show closure under subalgebras, let $A$ be an algebra in $\mathbf{C}$, and $B$ be a subalgebra of $A$, in the sense that it is closed under the operations. Then if $F_B$ is the free object given by the set $B$, the inclusion $B \hookrightarrow A$ extends to a unique homomorphism $f : F_B \to A$. We claim that the image of $f$ is $B$, so that property 3 implies $B$ is a subobject of $A$. Clearly each $b \in B$ is $f$ applied to the primitive expression $b$, so $B \subseteq \text{im } f$. Now suppose $w \in F_B$. Then $w \in F_{B'}$ for some finite subset $B'$ of $B$. Assume $B'$ is labeled $\{b_1, b_2, \ldots, b_n\}$; then there is an bijection $F_n \to F_{B'}$ sending $x_i \to b_i$, and $w$ gets mapped to by some $\omega \in F_n = \Omega(n)$. By definition of the $\Omega$-structures, $w$ is the expression $(\omega b_1 b_2 \ldots b_n)$. Consequently, $f(w) = (\omega b_1 b_2 \ldots b_n)$, the evaluation of the expression. Since $B$ is a subalgebra of $A$ and $b_1, b_2, \ldots b_n \in B$, $f(w) \in B$ follows, and $\text{im } f \subseteq B$ so that $\text{im } f = B$.

Closure under homomorphic images follows immediately from property 4 [and the fact that $\mathbf{C}$ is structure-based]. Recall the first and foremost definition of a congruence relation given in Section 1.4.

Closure under products follows from property 2. In particular, the empty batch consisting of no objects has a product of $\mathbf{C}$, and hence $T(\Omega)$ is in $\mathbf{C}$. This completes the proof that $\mathbf{C}$ is the category for a variety. ■

At this point it is natural to ask if any of the above 5 properties are redundant. It turns out if $\mathbf{C}$ follows properties 1, 3, 4, 5 but not necessarily 2, it can be proved to be a variety. This is more difficult, though; see Exercise 5.

However, Exercises 1-3 show that none of properties 3, 4, 5 are redundant; there are structure-based concrete categories satisfying any two of those but not the third, as well as satisfying properties 1 and 2.

**EXERCISES**

1. Here is an example of an algebraic category which is not finitary. Define a **complete join-semilattice** to be a partially ordered set $X$ such that every subset has a least upper bound. For example, the closed interval $[0, 1] \subseteq \mathbb{R}$ is a complete join-semilattice [due to the least upper bound property]. However, $\mathbb{R}$ is *not* a complete join-semilattice, because $\mathbb{Z} \subseteq \mathbb{R}$ has no least upper bound.

   (a) For each subset $S$ of $X$, let $U(S)$ be the least upper bound of $S$. Show that $U(\emptyset)$ and $U(X)$ are smallest and largest elements of $X$, respectively.

   (b) Then prove that for $x \in X$, $U(\{x\}) = x$ and for a set $\{S_\alpha\}$ of subsets, $U(\bigcup S_\alpha) = U(\{U(S_\alpha)\})$.

   (c) Define a **homomorphism** of complete join-semilattices $X, Y$ to be a map $f : X \to Y$ such that for every subset $S$ of $X$, $f(U(S)) = U(f(S))$. The complete join-semilattices and homomorphisms then form a structure-based concrete category. Show that this is an algebraic category which is not finitary. [*Hint*: For any set $Z$, $\mathcal{P}(Z)$ is a complete join-semilattice under inclusion. Show that along with $i : Z \to \mathcal{P}(Z)$ sending $i(z) = \{z\}$, it is a free object given by $Z$.]

2. If $X$ and $Y$ are partially ordered sets, define a map $f : X \to Y$ to be **order-preserving** if $x \leq y$ in $X$ implies $f(x) \leq f(y)$ in $Y$. Then the category of posets with order-preserving maps satisfies properties 1, 2, 4 and 5 in the definition of a finitary algebraic category, but not property 3. [*Hint*: The free object given by a set $X$ is the poset $X$ where $x \leq y$ means $x = y$.]

3. Define a **torsion-free abelian group** to be an abelian group in which every element except the identity has infinite order. Show that the full subcategory of **Ab** consisting of the torsion-free abelian groups satisfies properties 1, 2, 3 and 5 in the definition of a finitary algebraic category, but not property 4.

4. Let **C** be a finitary algebraic category. Do not use Theorem 2.8 to prove the following statements, because they are used in the proof of that theorem.

   (a) For any set $X$, let $F_X$ be the free object given by $X$. If $X'$ is a finite subset of $X$, explain why there is a unique morphism $F_{X'} \to F_X$ extending the inclusion map $X' \hookrightarrow X$.

   (b) Show that this morphism is injective, and identify elements of $F_{X'}$ with their images in $F_X$. Then $F_{X'}$ is a subobject of $F_X$.

   (c) For each $a \in F_X$, there exists a finite subset $X'$ of $X$ such that $a \in F_{X'}$. [This is the primary statement using property 5!]

5. Let **C** be a structure-based concrete category satisfying properties 1, 3, 4 and 5 in the definition of a finitary algebraic category.

   (a) Apply Theorem 2.8 to show that **C** is a class of $\Omega$-algebras for some signature $\Omega$, with all homomorphisms between them.

   (b) Show that the free objects all satisfy the same equational identities in their generating symbols.

   (c) Now show that **C** is the variety of $\Omega$-algebras satisfying those identities.

# 2.11 - Monads

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

**This section is not a prerequisite of any other and may be skipped if desired.**

Recall what expressions are in universal algebra. They comprise the free algebras, and are the most "abstract" way to apply operations to arbitrary symbols. The three fundamental things about an expression are that:

(1) Each symbol is in some way a primitive expression.

(2) An expression of expressions can be "flattened" into a great expression.

(3) Symbol substitutions can be made anywhere in the expressions.

And there are two additional facts of coherence:

(4) Given an expression of expressions of expressions [triple-layer!], it doesn't make a difference whether you flatten them starting from the inner layers or the outer layers.

(5) A primitive expression consisting of one expression flattens into the expression, and flattening an expression of primitive expressions yields the same expression with the symbols.

To see what those statements mean, let $\mathcal{V}(S)$ be a variety, then define $T : \mathbf{Set} \to \mathbf{Set}$ as follows. For each set $X$, define $TX$ to be the set $F_S(\Omega, X)$. Then every set map $f : X \to Y$ yields a unique homomorphism $\varphi : F_S(\Omega, X) \to F_S(\Omega, Y)$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ F_S(\Omega, X) & \xrightarrow{\varphi} & F_S(\Omega, Y) \end{array}$$

is commutative, so define $T(f)$ to be $\varphi$ regarded as a set map. It is clear then that $T$ is a functor. $T(f)$ does the job of "substituting symbols" in each expression in $X$, by where the map $f$ sends them to $Y$.

Now for each $X$, let $\eta_X = i_X : X \to TX$. The above diagram and how $T$ is defined immediately implies that $\eta$ is a natural transformation $1_{\mathbf{Set}} \Rightarrow T$. $\eta$ is the device which "sees each symbol as a primite expression," sending each element of $X$ to the length-one expression in $F_S(\Omega, X)$.

Next, notice that if $A$ is a $\mathcal{V}(S)$-algebra, the identity map $1_A : A \to A$, where the first $A$ is regarded as a set and the second as an algebra, extends to a unique homomorphism $e_A : F_S(\Omega, A) \to A$. It is called the **evaluation map** of $A$ and assigns every expression in $A$ to the value $A$ gives it. It is clear that for every homomorphism $f : A \to B$,

$$\begin{array}{ccc} F_S(\Omega, A) & \xrightarrow{T(f)} & F_S(\Omega, B) \\ {\scriptstyle e_A}\downarrow & & \downarrow{\scriptstyle e_B} \\ A & \xrightarrow{f} & B \end{array}$$

is commutative due to $f$ being a homomorphism.

Since the above holds for *any* algebras $A$ and $B$ and homomorphism $f$, it holds if $A = F_S(\Omega, X), B = F_S(\Omega, Y)$ and $f$ is the homomorphism $\varphi$ mentioned above. It follows that if each $\mu_X$ is assigned to be $e_{F_S(\Omega,X)} : F_S(\Omega, F_S(\Omega, X)) \to F_S(\Omega, X)$, $\mu$ is a natural transformation $TT \Rightarrow T$. $\mu$ is the device which takes each expression of expressions, and evaluates the expression in the free algebra, thus "flattening" the expression into one great expression.

Thus we have a triple $(T, \eta, \mu)$ where $\eta$ devices fact (1) about an expression, $\mu$ devices fact (2), and the functorial nature of $T$ devices fact (3). [Here's a nice little exercise: informally, what does the naturality of $\eta$ and $\mu$ say about the expressions?]

Now, note that for any algebra $A$, the fact that $e_A$ is a *homomorphism* implies that

$$\begin{array}{ccc} TTA & \xrightarrow{T(e_A)} & TA \\ {\scriptstyle e_{TA}}\downarrow & & \downarrow{\scriptstyle e_A} \\ TA & \xrightarrow{e_A} & A \end{array}$$

is commutative. Taking $TX$ for $A$ and noting that $\mu_X = e_{TX}$, the diagram becomes

$$\begin{array}{ccc} TTTX & \xrightarrow{T(\mu_X)} & TTX \\ {\scriptstyle \mu_{TX}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow{\mu_X} & TX \end{array}$$

This means $\mu_X T(\mu_X) = \mu_X \mu_{TX}$ for every set $X$; that is, the coherence condition $\mu(T\mu) = \mu(\mu T)$ holds, using the notation from Exercise 8 of Section 3. This is an associativity law for expression flattening: an expression of expressions of expressions has one unique flatten into an expression of the symbols.

It is also clear that $e_A \eta_A = 1_A$ for any algebra $A$, because $e_A$ extends $1_A$ using the universal property. Taking $TX$ for $A$, this becomes $\mu_X \eta_{TX} = 1_{TX}$ for every set $X$; that is, $\mu(\eta T) = 1_T$. Also, since $T(\eta_X) : TX \to TTX$ and $\mu_X : TTX \to TX$ are both homomorphisms, so is $\mu_X T(\eta_X) : TX \to TX$. Because $\mu_X T(\eta_X)$ clearly sends every symbol in $X$ to itself, $\mu_X T(\eta_X) = 1_{TX}$, so that also $\mu(T\eta) = 1_T$.

The statement $\mu(T\mu) = \mu(\mu T)$ mathematically states fact (4) and $\mu(T\eta) = 1_T = \mu(\eta T)$ states fact (5).

All of this leads to the following definition, which enables "expressions" to be natured on objects of an arbitrary category.

**DEFINITION**

*A* ***monad*** *is a triple* $(T, \eta, \mu)$ *where* $T : \mathbf{C} \to \mathbf{C}$ *is a functor and* $\eta : 1_{\mathbf{C}} \Rightarrow T$ *and* $\mu : TT \Rightarrow T$ *are natural transformations, such that* $\mu(T\mu) = \mu(\mu T)$ *and* $\mu(T\eta) = 1_T = \mu(\eta T)$*.* $\eta$ *is called the* ***unit*** *of the monad, and* $\mu$ *is called its*

***operator***.

**EXAMPLES**

1. We have shown above that any variety $\mathcal{V}(S)$ in universal algebra induces a monad $(T, \eta, \mu)$ on **Set** such that $TX = F_S(\Omega, X)$ for every set $X$. In general, for any takeoff from $\mathcal{V}(S_1)$ to $\mathcal{V}(S_2)$, there is a monad $(T, \eta, \mu)$ on $\mathbf{V}(S_2)$ which sends every $\mathcal{V}(S_2)$-algebra to the universal enveloping $\mathcal{V}(S_1)$-algebra, rasterized as a $\mathcal{V}(S_2)$-algebra. $\eta$ assigns each $B \in \mathcal{V}(S_2)$ the usual inclusion homomorphism $B \to TB$ given by the universal, and $\mu$ assigns each $B$ to the evaluation retraction [see Exercise 4 of Section 3] $r_{TB} : TTB \to TB$ in $\mathcal{V}(S_1)$, induced by the universal enveloping the rasterized $\mathcal{V}(S_1)$-algebra $TB$.

2. Let $M$ be a fixed monoid. Then $M$ induces a monad $(T, \eta, \mu)$ on **Set** as follows: Define $TX = M \times X$ and for each $f : X \to Y$, assign $T(f) : TX \to TY$ to send $(m, x) \to (m, f(x))$. Then let $\eta$ and $\mu$ be given by the monoid structure of $M$; that is, $\eta_X : X \to TX$ sends $x \to (1, x)$ and $\mu_X : TTX \to TX$ sends $(m, (n, x)) \to (mn, x)$. The naturality of $\eta$ and $\mu$ is clear, and the coherence conditions [e.g. $\mu(T\mu) = \mu(\mu T)$] follow from the associativity and unit laws of the monoid. This seemingly basic example is the special case of Example 1 where $\mathcal{V}(S)$ is the variety of $M$-actions.

3. Define a monad $(T, \eta, \mu)$ on **Set** by assigning $TX = \mathcal{P}(X)$ and $T(f) : X \to Y$ the map giving the image of a subset. Then assign $\eta_X : X \to TX$ to send each $x \in X$ to the one-element subset $\{x\}$, and assign $\mu_X : TTX \to TX$ to send each set of subsets of $X$ to their union. It is not hard to show that this defines a monad.

An interesting thing about a monad is that every adjunction yields a monad and vice versa. Throughout this section, we will use the definition of an adjunction in terms of its unit and counit [see Exercise 5 of Section 8]. That is:

I. Functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$

II. Natural transformations $\delta : 1_{\mathbf{D}} \Rightarrow FG, \epsilon : GF \Rightarrow 1_{\mathbf{C}}$

III. $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$

Recall the identities in Exercise 8(c) of Section 3; they will be very useful here. The following result is yielded.

**THEOREM 2.9** *Let $(G, F, \delta, \epsilon)$ be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit $\delta$ and counit $\epsilon$. Then $(FG, \delta, F\epsilon G)$ is a monad on the category $\mathbf{D}$.*

Taking the adjunction given by a takeoff and its universal, for instance, gives us Example 1 above.

*Proof of Theorem 2.9.* Clearly, $T = FG : \mathbf{D} \to \mathbf{D}$ so that makes sense. $\eta = \delta$ is a natural transformation $1_{\mathbf{D}} \Rightarrow FG$, hence a natural transformation $1_{\mathbf{D}} \Rightarrow T$. Since $\epsilon : GF \Rightarrow 1_{\mathbf{C}}$, $\mu = F\epsilon G : F(GF)G \Rightarrow F1_{\mathbf{C}}G$; that is, $\mu : FGFG \Rightarrow FG$, so $\mu$ is a natural transformation $TT \Rightarrow T$. We now prove the coherence conditions:

$$\mu(T\mu) = (F\epsilon G)[(FG)(F\epsilon G)] = (F\epsilon G)(FGF\epsilon G) = F[\epsilon(GF\epsilon)]G$$

$$\mu(\mu T) = (F\epsilon G)[(F\epsilon G)(FG)] = (F\epsilon G)(F\epsilon GFG) = F[\epsilon(\epsilon GF)]G$$

Yet $\epsilon(GF\epsilon) = \epsilon(\epsilon GF)$ by Exercise 8(e) of Section 3, with $GF$ in place of $F$ and $G$, $1_{\mathbf{C}}$ in place of $F'$ and $G'$ and $\epsilon$ in place of $\zeta$ and $\eta$. Therefore, $\mu(T\mu) = \mu(\mu T)$.

To show that $\mu(T\eta) = 1_T = \mu(\eta T)$:

$$\mu(T\eta) = (F\epsilon G)(FG\delta) = F[(\epsilon G)(G\delta)] = F1_G = 1_{FG} = 1_T$$

$$\mu(\eta T) = (F\epsilon G)(\delta FG) = [(F\epsilon)(\delta F)]G = 1_F G = 1_{FG} = 1_T$$

This completes the proof of the theorem. ■

## The Eilenberg-Moore category

You probably suspected that once expressions are defined, there's a general way to give an object "expression assignments" to make it an algebra. There certainly is. Also, there is a way to say a morphism of objects "preserves" the expression assignments. We now make this rigorous.

Recall the evaluation map $e_A : F_S(\Omega, A) \to A$ for $A \in \mathcal{V}(S)$. If $i_A : A \to F_S(\Omega, A)$ is the inclusion map, then $e_A i_A = 1_A$, because $e_A$ extends the set map $1_A$. This illustrates that $e_A$ assigns primitive expressions to their symbols, which is genuinely required of an expression evaluation.

Another important thing about $e_A$ is that if you have an expression of expressions in $A$, evaluating all the inner expressions then evaluating the resulting expression of symbols gives the same result as flattening the expression and then evaluating. This is, in effect, illustrated by the fact that $e_A$ is a homomorphism. In symbols, $e_A e_{F_S(\Omega,A)} = e_A \tilde{e}_A$, where $\tilde{e}_A$ is the homomorphism $F_S(\Omega, F_S(\Omega, A)) \to F_S(\Omega, A)$ extending the set map $i_A e_A : F_S(\Omega, A) \to F_S(\Omega, A)$.

Now consider the evaluation homomorphisms of two algebras, $e_A : F_S(\Omega, A) \to A$ and $e_B : F_S(\Omega, B) \to B$. If $f : A \to B$ is any function of the sets, $i_B f : A \to F_S(\Omega, B)$, has a unique extension to a homomorphism $\tilde{f} : F_S(\Omega, A) \to F_S(\Omega, B)$, and this $\tilde{f}$ makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ F_S(\Omega, A) & \xrightarrow{\tilde{f}} & F_S(\Omega, B) \end{array}$$

commutative. We claim that $f$ is a homomorphism if and only if *this* diagram

$$\begin{array}{ccc} F_S(\Omega, A) & \xrightarrow{\tilde{f}} & F_S(\Omega, B) \\ {\scriptstyle e_A}\downarrow & & \downarrow{\scriptstyle e_B} \\ A & \xrightarrow{f} & B \end{array}$$

is commutative. For, if the diagram is commutative, $f$ is the result of injectifying the homomorphism $e_B \tilde{f}$ over the quotient map $e_A$, and is therefore a homomorphism. Conversely, if $f$ is a homomorphism, then for all $\omega \in$

$\Omega(n), a_1, a_2, \dots, a_n \in A$,

$$e_B\tilde{f}(\overline{\omega}a_1a_2\dots a_n) = e_B(\overline{\omega}f(a_1)f(a_2)\dots f(a_n)) = (\omega f(a_1)f(a_2)\dots f(a_n))$$

$$= e_B(\omega f(a_1)f(a_2)\dots f(a_n)) = e_B\tilde{f}(\omega a_1a_2\dots a_n)$$

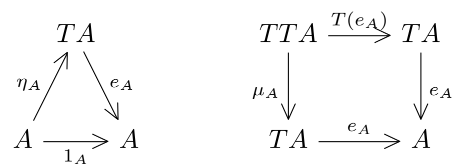
where the $\overline{\omega}$'s give the unevaluated expression in the free algebra given by the set. It follows that $\ker e_A \subseteq \ker(e_B\tilde{f})$ and that the injectification of $e_B\tilde{f} = ge_A$ for some homomorphism $g$. Then, $g = g1_A = ge_Ai_A = e_B\tilde{f}i_A = e_Bi_Bf = 1_Bf = f$ so $g = f$ and the statement of the diagram, $e_B\tilde{f} = fe_A$ holds.

To get a better understanding, note that one pair of arrows in the above diagram sends all the symbols in the expression over the map $f$, then evaluates the resulting expression, whereas the other pair evaluates the expression then sends it over $f$. The virtue of $f$ being a homomorphism is that there's no difference between those.

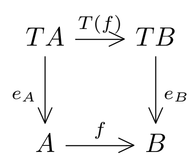
We now define the Eilenberg-Moore category.

**DEFINITION**

*Let $(T, \eta, \mu)$ be a monad. An* ***algebra*** *for $(T, \eta, \mu)$ is a pair $(A, e_A)$ with $A \in ob(\mathbf{C})$, $e_A : TA \to A$ such that $e_A\eta_A = 1_A$ and $e_A\mu_A = e_AT(e_A)$:*




*If $(A, e_A)$ and $(B, e_B)$ are algebras for $(T, \eta, \mu)$, an* ***algebra morphism*** *$f : (A, e_A) \to (B, e_B)$ is a morphism $f : A \to B$ in $\mathbf{C}$ satisfying $e_BT(f) = fe_A$:*




*The* ***Eilenberg-Moore category*** *given by the monad is defined to be the category of algebras whose morphisms are algebra morphisms, with the usual composition and identity morphisms. This category is denoted $\mathbf{C}^T$.*

It is immediate that for any $X \in \mathbf{ob}(\mathbf{C})$, $(TX, \mu_X)$ is an algebra, and if $(A, e_A)$ is an algebra, then $e_A : (TA, \mu_A) \to (A, e_A)$ is an algebra morphism. Also, for any morphism $f : X \to Y$, $T(f)$ is an algebra morphism $(TX, \mu_X) \to (TY, \mu_Y)$ because of the naturality of $\mu$.

**EXAMPLES**

1. Let $(T, \eta, \mu)$ be the monad on **Set** given by a variety $\mathcal{V}(S)$. We have seen that a $\mathcal{V}(S)$ algebra $A$, along with its evaluation homomorphism $e_A$ :

$F_S(\Omega, A) \to A$, is an algebra for $(T, \eta, \mu)$. Conversely, any algebra for the monad is a $\mathcal{V}(S)$ algebra in a clean, deterministic way. Also, algebra morphisms are simply homomorphisms in $\mathbf{V}(S)$, making the Eilenberg-Moore category identifiably $\mathbf{V}(S)$.

2. Let $(T, \eta, \mu)$ be the monad on **Set** with $TX = \mathcal{P}(X)$ as in Example 3. We wish to find its Eilenberg-Moore category. To begin with, an algebra for $(T, \eta, \mu)$ is a set $A$ with a map $e_A : \mathcal{P}(A) \to A$ such that $e_A \eta_A = 1_A$ and $e_A \mu_A = e_A T(e_A)$. Using the definition of $\eta$ and $\mu$ for this monad, this says $e_A(\{a\}) = a$ for $a \in A$ and $e_A(\bigcup S_\alpha) = e_A(\{e_A(S_\alpha)\})$ for subsets $S_\alpha$ of $X$.

If $a, b \in A$, define $a \leq b$ to mean $e_A(\{a, b\}) = b$. We claim that $A$ is a complete join-semilattice with $e_A$ giving the least upper bound of any set. $e_A(\{a, a\}) = e(\{a\}) = a$, so $a \leq a$, proving reflexivity. If $a \leq b$ and $b \leq a$, then $e_A(\{a, b\})$ is simultaneously $a$ and $b$, whence $a = b$, proving antisymmetry. Now suppose $a \leq b$ and $b \leq c$. Then $e_A(\{a, b\}) = b$ and $e_A(\{b, c\}) = c$. Consequently,

$$\begin{aligned}
&e_A(\{a, c\}) \\
&= e_A(\{\ e_A(\{a\}),\ e_A(\{b, c\})\ \}) && \text{(because } e_A(\{a\}) = a \text{ and } e_A(\{b, c\}) = c) \\
&= e_A(\{a\} \cup \{b, c\}) && \text{(because } e_A(\{e_A(S_\alpha)\}) = e_A(\textstyle\bigcup S_\alpha)) \\
&= e_A(\{a, b, c\}) \\
&= e_A(\{a, b\} \cup \{c\}) \\
&= e_A(\{\ e_A(\{a, b\}),\ e_A(\{c\})\ \}) && \text{(because } e_A(\{e_A(S_\alpha)\}) = e_A(\textstyle\bigcup S_\alpha)) \\
&= e_A(\{b, c\}) && \text{(because } e_A(\{a, b\}) = b \text{ and } e_A(\{c\}) = c) \\
&= c.
\end{aligned}$$

Therefore, $e_A(\{a, c\}) = c$ and $a \leq c$, proving transitivity. Hence $(A, \leq)$ is a poset. Now we show that for any subset $X$, $e_A(X)$ is the least upper bound of $X$. Suppose $x \in X$; then $e_A(\{x, e_A(X)\}) = e_A(\{\ e_A(\{x\}),\ e_A(X)\ \}) = e_A(\{x\} \cup X) = e_A(X)$, proving that $x \leq e_A(X)$. Therefore $e_A(X)$ is an upper bound of $X$. Now suppose $y$ is any upper bound of $X$. If $X \neq \emptyset$, then set theory shows

$$X \cup \{y\} = \bigcup_{x \in X} \{x, y\}$$

So $e_A(\{e_A(X), y\}) = e_A(\{\ e_A(X),\ e_A(\{y\})\ \}) = e_A(X \cup \{y\}) = e_A(\bigcup_{x \in X} \{x, y\}) = e_A(\{e_A(\{x, y\}) \mid x \in X\})$. Since $y$ is an upper bound of $X$, every $x \in X$ satisfies $x \leq y$ so that $e_A(\{x, y\}) = y$. Furthermore, $e_A(\{e_A(\{x, y\}) \mid x \in X\}) = e_A(\{y \mid x \in X\}) = e_A(\{y\}) = y$. If $X = \emptyset$, then $e_A(\{e_A(X), y\}) = e_A(X \cup \{y\}) = e_A(\{y\}) = y$. Thus $e_A(\{e_A(X), y\}) = y$, which means that $e_A(X) \leq y$ and $e_A(X)$ is the *least* upper bound of $X$.

This proves that any algebra $A$ for the monad is a complete join-semilattice, which $e_A$ giving the least upper bound. Exercise 1 of Section 10 shows the converse: if $A$ is a complete join-semilattice and $e_A : \mathcal{P}(A) \to A$ gives the least upper bound, then $e_A(\{a\}) = a$ and $e_A(\bigcup S_\alpha) = e_A(\{e_A(S_\alpha)\})$; therefore, $A$ is an algebra for the monad. It is then clear that morphisms are complete join-semilattice homomorphisms, and that the Eilenberg-Moore Category of $(T, \eta, \mu)$ is the category of complete join-semilattices.

We now show one way to get an adjunction from any monad; Exercise 5 shows another way.

**THEOREM 2.10** *Let $(T, \eta, \mu)$ be a monad on $\mathbf{C}$, and let $\mathbf{C}^T$ be its Eilenberg-Moore category. Define $F, G, \delta, \epsilon$ as follows:*

*$F : \mathbf{C}^T \to \mathbf{C}$ sends an algebra $(A, e_A) \to A$ and a morphism $h : (A, e_A) \to (B, e_B)$ to the morphism $h : A \to B$ in $\mathbf{C}$.*

*$G : \mathbf{C} \to \mathbf{C}^T$ sends an object $X \to (TX, \mu_X)$ and a morphism $f : X \to Y$ to the morphism $T(f) : TX \to TY$ of $\mathbf{C}^T$.*

*For each $X \in ob(\mathbf{C})$, $\delta_X : X \to FGX$ is assigned to be $\eta_X : X \to TX$.*

*For each $(A, e_A) \in \mathbf{C}^T$, $\epsilon_A : GFA \to A$ is assigned to be $e_A : (TA, \mu_A) \to (A, e_A)$.*

*Then $\delta : 1_{\mathbf{C}} \Rightarrow FG$ and $\epsilon : GF \Rightarrow 1_{\mathbf{C}^T}$ are natural transformations and $G, F$ are adjoint functors with unit $\delta$ and counit $\epsilon$.*

*Proof of Theorem 2.10.* Notice that $FG$ is the functor $T$, and that $\delta$ is the same as $\eta$. Therefore, $\delta$ is a natural transformation. The naturality of $\epsilon$ follows from the definition of a morphism in $\mathbf{C}^T$. It suffices to show that $(F\epsilon)(\delta F) = 1_F$ and $(\epsilon G)(G\delta) = 1_G$, then we have an adjunction.

Take each $(A, e_A) \in \text{ob}(\mathbf{C}^T)$. Then $[(F\epsilon)(\delta F)]_A = F(\epsilon_A)\delta_{FA} = e_A\eta_{FA} = 1_{FA}$, because $FA$ is simply $A$ without its $e_A$ equipment. Therefore, $(F\epsilon)(\delta F) = 1_F$.

Now take each $X \in \text{ob}(\mathbf{C})$. Then $[(\epsilon G)(G\delta)]_X = \epsilon_{GX}G(\delta_X) = \epsilon_{(TX,\mu_X)}G(\eta_X) = \mu_X T(\eta_X) = [\mu(T\eta)]_X = (1_T)_X = 1_{TX} = 1_{GX}$. Hence $(\epsilon G)(G\delta) = 1_G$. ■

**EXERCISES**

1. Let $\mathbf{C}$ be the category of complete join-semilattices [Exercise 1 of Section 10], $F : \mathbf{C} \to \mathbf{Set}$ the forgetful functor and $G : \mathbf{Set} \to \mathbf{C}$ the free-object giving functor. Then $G$ and $F$ are adjoint functors. Show that the monad induced by them in Theorem 2.9 is the monad on **Set** in Example 3.

2. A **closure operator** on a poset $X$ is a function $C : X \to X$ satisfying the following laws for all $x, y \in X$:

   *Extension*: $x \leq C(x)$;

   *Idempotence*: $C(C(x)) = C(x)$;

   *Monotonic Increase*: If $x \leq y$, then $C(x) \leq C(y)$.

   For example, if $A \in \mathcal{V}(S)$, "subalgebra generated by" and "congruence relation generated by" are closure operators on $\mathcal{P}(A)$ and $\mathcal{P}(A \times A)$, respectively.

   Show that if $X$ is regarded as a category with at most one morphism in each hom-set [see Example 7 of Section 1], a monad on $X$ is identifiably a closure operator.

3. A comonad on a category $\mathbf{C}$ is a triple $(T, \eta, \mu)$ with $\eta : T \Rightarrow 1_{\mathbf{C}}$, $\mu : T \Rightarrow TT$ natural transformations, such that $(T\mu)\mu = (\mu T)\mu$ and $(T\eta)\mu = 1_T = (\eta T)\mu$ hold.

(a) A comonad on $\mathbf{C}$ is identifiably a monad on $\mathbf{C}^{\mathrm{op}}$.

(b) Let $(G, F, \delta, \epsilon)$ be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit $\delta$ and counit $\epsilon$. Then $(GF, \epsilon, G\delta F)$ is a comonad on $\mathbf{C}$.

(c) Use part (b) to obtain a comonad on $\mathbf{V}(S)$.

4. (a) If you start with a monad, take its induced adjunction in Theorem 2.10 involving the Eilenberg-Moore category, then take the adjunction's induced monad in Theorem 2.9, show that you have the same monad you started with.

   (b) If you start with an adjunction, take its induced monad [Theorem 2.9] and then the resulting adjunction [Theorem 2.10], then you *don't* necessarily arrive at the same adjunction you started with up to equivalence.

   If the adjunction's "typical" enough that you *would* revisit the same adjunction when doing this, the adjunction is said to be **monadic** [or tripleable].

   (c) Prove that the adjunction given by a takeoff is monadic. [It is best to start off with takeoffs from a variety to the sets.]

5. Let $(T, \eta, \mu)$ be a monad on $\mathbf{C}$, and $\mathbf{C}^T$ be its Eilenberg-Moore category. Define a new category $\mathbf{C}_T$ by $\mathrm{ob}(\mathbf{C}_T) = \mathrm{ob}(\mathbf{C})$ and for $X, Y \in \mathrm{ob}(\mathbf{C}_T)$, $\hom_{\mathbf{C}_T}(X, Y) = \hom_{\mathbf{C}^T}((TX, \mu_X), (TY, \mu_Y))$. Define composition of morphisms and identity morphisms to agree with $\mathbf{C}^T$. Then $\mathbf{C}_T$ is called the **Kleisli category** for the monad.

   (a) Show that there is a bijection between $\hom_{\mathbf{C}_T}(X, Y)$ and $\hom_{\mathbf{C}}(X, TY)$ for all $X, Y \in \mathrm{ob}(\mathbf{C})$. [*Hint*: If $f \in \hom_{\mathbf{C}^T}((TX, \mu_X), (TY, \mu_Y))$, consider $f\eta_X : X \to TY$. The other way around, for each $g : X \to TY$, show that $\mu_Y T(g)$ is in $\mathbf{C}^T$.]

   (b) Define $F, G, \delta, \epsilon$ as follows:

   $F : \mathbf{C}_T \to \mathbf{C}$ sends each $X \in \mathrm{ob}(\mathbf{C}_T)$ to $TX$, and each $f \in \hom_{\mathbf{C}_T}(X, Y)$ to $f$ itself as a morphism $TX \to TY$.

   $G : \mathbf{C} \to \mathbf{C}_T$ sends each $X \in \mathrm{ob}(\mathbf{C})$ to $X$ regarded as an object in $\mathbf{C}_T$, and each $f \in \hom_{\mathbf{C}}(X, Y)$ to $T(f) : TX \to TY$ in $\hom_{\mathbf{C}_T}(X, Y)$.

   For each $X \in \mathrm{ob}(\mathbf{C})$, $\delta_X : X \to FGX$ is assigned to be $\eta_X : X \to TX$.

   For each $Y \in \mathrm{ob}(\mathbf{C}_T)$, $\epsilon_Y : GFY \to Y$ is assigned to be $\mu_Y \in \hom_{\mathbf{C}_T}(TY, Y)$.

   Show that $\delta$ and $\epsilon$ are natural transformations, and that $G$ and $F$ are adjoint functors with unit $\delta$ and counit $\epsilon$.

   (c) Prove or disprove: If you use Theorem 2.9 to turn this adjunction into a monad on $\mathbf{C}$, you necessarily get the same monad you started with.

6. Let $(G, F, \delta, \epsilon)$ be an adjunction with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, unit $\delta$ and counit $\epsilon$. Then let $(T, \eta, \mu)$ be the monad on $\mathbf{D}$ induced by Theorem 2.9; that is, $(FG, \delta, F\epsilon G)$.

(a) For each $A \in \mathbf{C}$, $(FA, F(\epsilon_A))$ is an algebra for $(T, \eta, \mu)$.

(b) For each $f : A \to A'$ in $\mathbf{C}$, $F(f)$ is an algebra morphism $(FA, F(\epsilon_A)) \to (FA', F(\epsilon_{A'}))$.

(c) Now suppose $(G', F', \delta', \epsilon')$ is the adjunction between $\mathbf{D}^T, \mathbf{D}$ established in Theorem 2.10. Show that there is a unique functor $E : \mathbf{C} \to \mathbf{D}^T$ such that $F = F'E$ and $EG = G'$. Also note that $\delta = \delta'$ because both are equal to $\eta$.

(d) Suppose now that $(G', F', \delta', \epsilon')$ is the adjunction between the Kleisli category $\mathbf{D}_T$ and $\mathbf{D}$ in Exercise 5. Define $K : \mathbf{D}_T \to \mathbf{C}$ by $KB = GB$ for $B \in \mathrm{ob}(\mathbf{D}_T)$ [$= \mathrm{ob}(\mathbf{D})$], and $K(f) = \epsilon_{GB'} G(f\eta_B)$ for morphisms $f \in \hom_{\mathbf{D}^T}(TB, TB')$ [$= \hom_{\mathbf{D}_T}(B, B')$]. Then $K$ is the unique functor $\mathbf{D}_T \to \mathbf{C}$ such that $F' = FK$ and $KG' = G$.

7. Suppose $(T, \eta, \mu)$ is a monad on a category $\mathbf{D}$. Define a category $\mathbf{Adj}(\mathbf{D}, T)$ as follows:

   (1) Objects are adjunctions $(G, F, \delta, \epsilon)$ with $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ where $\mathbf{C}$ is any category, and the induced monad $(FG, \delta, F\epsilon G)$ is the given monad $(T, \eta, \mu)$.

   (2) Morphisms are morphisms of adjunctions [see Exercise 8 of Section 8] which are the identity on $1_{\mathbf{D}}$.

   (a) Verify that this is indeed a category.

   (b) The adjunction involving the Eilenberg-Moore category [Theorem 2.10] is a terminal object. [*Hint*: Use Exercise 6.]

   (c) The adjunction involving the Kleisli category [Exercise 5] is an initial object.

# 2.12 - Monads on **Set**

Nicholas McConnell

(Categories)

*The material and exposition for this lesson follows an imaginary textbook on Dozzie Abstract Algebra.*

**This section is not a prerequisite of any other and may be skipped if desired.**

In Section 10, we classified algebraic categories, and showed that each one is isomorphic to some variety in universal algebra. In this section we will do something surprisingly easier: we will take an arbitrary monad $(T, \eta, \mu)$ on the category **Set**, and show that it is the monad sending $X \to F_S(\Omega, X)$ for some variety $\mathcal{V}(S)$. This, in effect, associates any adjunction $(G, F, \eta)$ with $F : \mathbf{C} \to \mathbf{Set}$ with a monadic adjunction involving a variety.

There is one obstacle of this, though: the variety might have infinitary operations if we are not careful. For example, the least upper bound in a complete join-semilattice is infinitary — it allows infinitely many operands at once. We have always assumed varieties consist of *finitary* operations, so we must first define a condition on the monad.

**DEFINITION** *A monad $(T, \eta, \mu)$ on* **Set** *is said to be* ***finitary*** *provided that whenever $X$ is a set and $w \in TX$, there exists a finite subset $X'$ of $X$ such that if $\iota : X' \to X$ is the inclusion map, then $T(\iota) : TX' \to TX$ has $w$ in its image.*

This statement says that any expression in $TX$ uses only finitely many symbols. Thus it guarantees the "finiteness" of expressions and operators, and is related to Lemma 1.20 in the previous chapter. We now state and prove our theorem.

**THEOREM 2.11** *If $(T, \eta, \mu)$ is a finitary monad on* **Set***, then there exists a variety $\mathcal{V}(S)$ such that $(T, \eta, \mu)$ is the monad sending $X \to F_S(\Omega, X)$, in Example 1 of Section 11.*

(The theorem also holds if the monad is not finitary, but then there would be infinitary operators in $\mathcal{V}(S)$, which is beyond our course.)

*Proof of Theorem 2.11.* First we define a universal-algebra signature $\Omega$. For each $n \geq 0$, set $\Omega(n) = T\{x_1, x_2, \ldots, x_n\}$. In particular, $\Omega(0) = T\emptyset$.

Now, for each set $X$ we form the following $\Omega$-algebra structure on $TX$: For $\omega \in \Omega(n)$, $a_1, a_2, \ldots, a_n \in TX$, let $\varphi$ be the map $x_i \to a_i$ from $\{x_1, x_2, \ldots, x_n\} \to TX$. Then $\mu_X T(\varphi)$ goes from $\Omega(n) \to TX$; define $(\omega a_1 a_2 \ldots a_n) = \mu_X T(\varphi)(\omega)$. In particular, if $n = 0$, let $\varphi$ be the unique map $\emptyset \to TX$ and define $(\omega_{TX}) = \mu_X T(\varphi)(\omega)$. We thus have made each $TX$ into an $\Omega$-algebra. It is evident that if $\omega \in \Omega(n)$ then $(\omega\eta_X(x_1)\eta_X(x_2)\ldots\eta_X(x_n))$ is the element $\omega$ of $T\{x_1, x_2, \ldots, x_n\}$.

We now form a set $S \subseteq F(\Omega, X_0)^2$ of identities for a variety $\mathcal{V}(S)$. [Recall that $X_0$ is a countably infinite set.] For any expressions $w_1, w_2 \in F(\Omega, X_0)$, we have $w_1, w_2 \in F(\Omega, X')$ for some finite subset of $X' = \{x_1, x_2, \ldots, x_n\}$ of $X_0$. Thus $w_1$ and $w_2$ are expressions in $x_1, x_2, \ldots, x_n$, and therefore, when each $x_i$

is changed to $\eta_{X'}(x_i)$ the resulting expressions can be evaluated to elements of the algebra $TX' = \Omega(n)$; if they are the same element, assign $(w_1, w_2) \in S$.

We now claim that $\mathcal{V}(S)$ is our desired variety; to do this, we must show four things:

(1) For any set $X$, $TX$ is identifiably $F_S(\Omega, X)$;
(2) For any $f : X \to Y$; $T(f) : TX \to TY$ is the homomorphism $f : F_S(\Omega, X) \to F_S(\Omega, Y)$ which extends $f$;
(3) $\eta_X : X \to TX$ is the usual inclusion $X \to F_S(\Omega, X)$;
(4) $\mu_X : TTX \to TX$ is the evaluation homomorphism $F_S(\Omega, TX) \to TX$.

This will mean the monad necessarily matches up with the one we have in the previous section.

We first show that for each $f : X \to Y$, $T(f) : TX \to TY$ is a homomorphism. Take any $\omega \in \Omega(n), a_1, a_2, \ldots, a_n \in TX$. Let $g_1 : \{x_1, x_2, \ldots, x_n\} \to TX$ sending $x_i \to a_i$, so $g_2 = T(f)g_1 : \{x_1, x_2, \ldots, x_n\} \to TY$ sends $x_i \to T(f)(a_i)$. Then by definition of $\omega$, $\mu_X T(g_1)(\omega) = (\omega a_1 a_2 \ldots a_n)$, and $\mu_Y T(g_2)(\omega) = (\omega T(f)(a_1) T(f)(a_2) \ldots T(f)(a_n))$. However, $\mu_Y T(g_2) = \mu_Y T(T(f)g_1) = \mu_Y TT(f)T(g_1) = T(f)\mu_X T(g_1)$, by naturality of $\mu$. Thus $\mu_Y T(g_2)(\omega) = T(f)[\mu_X T(g_1)(\omega)] = T(f)(\omega a_1 a_2 \ldots a_n)$, and

$$T(f)(\omega a_1 a_2 \ldots a_n) = (\omega T(f)(a_1) T(f)(a_2) \ldots T(f)(a_n)),$$

proving that $T(f)$ is a homomorphism.

Next, we prove that $\mu_X : TTX \to TX$ is a homomorphism. For any $\omega \in \Omega(n)$, $a_1, a_2, \ldots, a_n \in TTX$, let $g_1 : \{x_1, x_2, \ldots, x_n\} \to TTX$ send $x_i \to a_i$ and $g_2 = \mu_X g_1 : \{x_1, x_2, \ldots, x_n\} \to TX$. Then by definition of $\omega$, $\mu_{TX} T(g_1)(\omega) = (\omega a_1 a_2 \ldots a_n)$, and $\mu_X T(g_2)(\omega) = (\omega \mu_X(a_1) \mu_X(a_2) \ldots \mu_X(a_n))$. But $\mu_X T(g_2) = \mu_X T(\mu_X g_1) = \mu_X T(\mu_X) T(g_1) = \mu_X \mu_{TX} T(g_1)$ [recall that $\mu(T\mu) = \mu(\mu T)$]. Therefore, $\mu_X T(g_2)(\omega) = \mu_X[\mu_{TX} T(g_1)(\omega)] = \mu_X(\omega a_1 a_2 \ldots a_n)$. It follows that $\mu_X$ is a homomorphism, as $\mu_X T(g_2)(\omega)$ is equal to both $\mu_X(\omega a_1 a_2 \ldots a_n)$ and $(\omega \mu_X(a_1) \mu_X(a_2) \ldots \mu_X(a_n))$.

We are now ready to prove statements (1)-(4) above.

To show (1), first we need to prove that the $\Omega$-algebra $TX$ is in $\mathcal{V}(S)$. To do this, take any homomorphism $\varphi : F(\Omega, X_0) \to TX$ and $(w_1, w_2) \in S$. Then $w_1, w_2 \in F(\Omega, X')$ for some finite subset $X' = \{x_1, x_2, \ldots, x_n\}$ of $X_0$. Furthermore, $w_1, w_2 \in \Omega(n)$, and if $a_i = \varphi(x_i) \in TX$ for each $i$, then $\varphi(w_j) = (w_j a_1 a_2 \ldots a_n)$ for $j = 1, 2$ because $\varphi$ is a homomorphism. Now take the map $\psi : X' \to TX$ sending $x_i \to a_i$ and form $\phi = \mu_X T(\psi) : TX' \to TX$. The last two paragraphs imply that $\phi$ is a homomorphism, so $\phi(w_j x_1 x_2 \ldots x_n) = (w_j a_1 a_2 \ldots a_n)$ for $j = 1, 2$. However, $(w_1 x_1 x_2 \ldots x_n)$ and $(w_2 x_1 x_2 \ldots x_n)$ are the same element of $TX'$, because $(w_1, w_2) \in S$. Therefore, $(w_1 a_1 a_2 \ldots a_n) = (w_2 a_1 a_2 \ldots a_n)$, from which it follows that $(w_1, w_2) \in F(\Omega, X_0)^2$ is in the kernel of $\varphi$. Therefore, $TX \in \mathcal{V}(S)$.

Because of this, there is a unique homomorphism $h_X : F_S(\Omega, X) \to TX$ extending the set map $\eta_X : X \to TX$. We claim that $h_X$ is an isomorphism. To support this claim, first suppose $X$ is a finite set, say $\{x_1, x_2, \ldots, x_n\}$. Then if $h_X(\bar{e}_1) = h_X(\bar{e}_2)$ with $e_1, e_2 \in F(\Omega, X)$, then after identifying $X$ with a subset of $X_0$, $e_1$ and $e_2$ are expressions in $x_1, x_2, \ldots, x_n$ which evaluate, after substituting $x_i \to \eta_X(x_i)$, to the same element in $TX = \Omega(n)$. Hence $(e_1, e_2) \in S$ by definition of $S$ and $\bar{e}_1 = \bar{e}_2$. Therefore, $h_X$ is injective. For each $\omega \in TX = \Omega(n)$, $h_X$ sends the expression $(\omega x_1 x_2 \ldots x_n)$ to $(\omega \eta_X(x_1)\eta_X(x_2)\ldots\eta_X(x_n)) = \omega$ (because $h_X$ is a homomorphism), so $h_X$ is surjective. Thus $h_X$ is an isomorphism if $X$ is finite.

Now suppose $X$ is infinite. For each finite subset $X'$ of $X$, recall that $F_S(\Omega, X')$ is a subalgebra of $F_S(\Omega, X)$, and realize that $TX'$ is identifiably a subalgebra of $TX$, because if $\iota : X' \to X$ is the inclusion map, $T(\iota)$ is injective, and is a homomorphism as proved above. It is clear that $h_X$ sends elements of $F_S(\Omega, X')$ to elements of $TX'$. Hence $h_X|F_S(\Omega, X')$ is the homomorphism from $F_S(\Omega, X') \to TX'$ sending $x \in X'$ to $\eta_X(x) = \eta_{X'}(x)$; by the argument in the previous paragraph, it is an isomorphism. Thus whenever $h_X(\bar{e}_1) = h_X(\bar{e}_2)$, the restriction of $h_X$ to $F_S(\Omega, X')$ with some suitable finite subset $X'$ sends $\bar{e}_1$ and $\bar{e}_2$ to the same element of $TX'$, so that $\bar{e}_1 = \bar{e}_2$ and $h_X$ is injective. The surjectivity of $h$ follows from the fact that $(T, \eta, \mu)$ is finitary — for all $\omega \in TX$, $\omega \in TX'$ for some finite subset $X'$ of $X$, and therefore $h$ sends an element of $F_S(\Omega, X')$ to $\omega$.

This proves that $h_X$ is an isomorphism, and it identifies $TX$ with $F_S(\Omega, X)$. The proof of (1) is complete.

(3) follows from the fact that if $i_X : X \to F_S(\Omega, X)$ is the usual free-algebra inclusion, then $h_X i_X = \eta_X$. Thus when $h_X$ identifies the algebras together, it identifies $i_X$ and $\eta_X$ together.

As for statement (2), we already know that $T(f) : TX \to TY$ is a homomorphism, so we need only show that $T(f)$ sends $\eta_X(x), x \in X$ to $\eta_Y(f(x))$. This is an immediate consequence of the naturality of $\eta$, which implies $T(f)\eta_X = \eta_Y f$.

Now to prove statement (4): we already proved that $\mu_X$ is a homomorphism, so we only need to show that it extends the identity map $TX \to TX$; that is, $\mu_X \eta_{TX} = 1_{TX}$. But this follows immediately from the monad axiom $\mu(\eta T) = 1_T$.

Thus statements (1)-(4) are proven and the proof is completed. ■

It's not easy to overestimate the power of what we have just proved. We've shown that from (almost) every monad on **Set**, we can recover a variety in universal algebra. There may be many other adjunctions from which monads on **Set** are formed; but those adjunctions would most likely not be monadic.

**EXERCISES**

1. Prove or disprove:

   (a) If $(T, \eta, \mu)$ is a monad on **Set**, then its Eilenberg-Moore category $\mathbf{C}^T$ is an algebraic category [Section 10].

   (b) $(T, \eta, \mu)$ is a finitary monad if and only if $\mathbf{C}^T$ is a finitary algebraic category.

2. Tell whether or not each monad on **Set** is finitary. If it is, state which variety it comes from. If not, find some other way to describe the Eilenberg-Moore category.

(a) $TX = \mathcal{P}(X)$; for $f : X \to Y$, $T(f)$ is the image map $\mathcal{P}(X) \to \mathcal{P}(Y)$ sending $S \to f(S)$; $\eta_X : X \to \mathcal{P}(X)$ sends each $x \in X$ to $\{x\}$; $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ sends each set of subsets of $X$ to the union of the subsets. [Example 3 of Section 11]

(b) Same as part (a), but $TX$ is the set of nonempty subsets of $X$.

(c) Same as part (a), but $TX$ is the set of finite subsets of $X$.

(d) Same as part (a), but $TX$ is the set of countable subsets of $X$.

(e) Let $M$ be a fixed monoid. $TX = M \times X$; for $f : X \to Y$, $T(f)$ is the map $M \times X \to M \times Y$ sending $(m, x) \to (m, f(x))$; $\eta_X : X \to M \times X$ sends each $x \in X$ to $(1, x)$; $\mu_X : M \times (M \times X) \to M \times X$ sends $(m_1, (m_2, x)) \to (m_1 m_2, x)$. [Example 2 of Section 11]

(f) $TX = \{\circ\}$ for all $X$ and $T(f), \eta, \mu$ are defined in the unique ways.

(g) $TX = X \times X$; for $f : X \to Y$, $T(f)$ is the map $X \times X \to Y \times Y$ sending $(x_1, x_2) \to (f(x_1), f(x_2))$; $\eta_X : X \to X \times X$ sends each $x \in X$ to $(x, x)$; $\mu_X : (X \times X) \times (X \times X) \to X \times X$ sends $((x_1, x_2), (x_3, x_4))$ to $(x_1, x_4)$.

(h) Let $S$ be a fixed set. $TX = X^S$ [functions from $S$ to $X$]; for $f : X \to Y$, $T(f)$ is the map $X^S \to Y^S$ sending $h \to fh$ for $h : S \to X$; $\eta_X : X \to X^S$ sends each $x \in X$ to the constant function $s \to x$ in $X^S$; $\mu_X : (X^S)^S \to X^S$ sends each $h : S \to X^S$ to the map $s \to h(s)(s)$ from $S \to X$. [*Caution*: Whether this monad is finitary depends on something about $S$. Parts (f) and (g) are special cases of this, so they may help.]

(i) Let $S$ be a fixed set. $TX = X \uplus S$; for $f : X \to Y$, $T(f)$ is the map $X \uplus S \to Y \uplus S$ sending each $x \in X$ to $f(x)$ and each element of $S$ to itself; $\eta_X : X \to X \uplus S$ sends each $x$ to itself in the disjoint union summand $X$; $\mu_X : (X \uplus S) \uplus S = X \uplus S$ sends each $x$ to itself and each $s$ in either of the $S$'s to itself in the summand $S$.